Many electric circuits are complex, but it is an engineer’s goal to reduce their complexity to analyze them easily. In the previous chapters, we have mastered the ability to solve networks containing independent and dependent sources making use of either mesh or nodal analysis. In this chapter, we will introduce new techniques to strengthen our armoury to solve complicated networks. Also, these new techniques in many cases do provide insight into the circuit’s operation that cannot be obtained from mesh or nodal analysis. Most often, we are interested only in the detailed performance of an isolated portion of a complex circuit. If we can model the remainder of the circuit with a simple equivalent network, then our task of analysis gets greatly reduced and simplified. For example, the function of many circuits is to deliver maximum power to load such as an audio speaker in a stereo system. Here, we develop the required relationship between a load resistor and a fixed series resistor which can represent the remaining portion of the circuit. Two of the theorems that we present in this chapter will permit us to do just that.

3.1 Superposition theorem

The principle of superposition is applicable only for linear systems. The concept of superposition can be explained mathematically by the following response and excitation principle:

\[ i_1 \rightarrow v_1 \]
\[ i_2 \rightarrow v_2 \]

then,
\[ i_1 + i_2 \rightarrow v_1 + v_2 \]

The quantity to the left of the arrow indicates the excitation and to the right, the system response. Thus, we can state that a device, if excited by a current \( i_1 \) will produce a response \( v_1 \). Similarly, an excitation \( i_2 \) will cause a response \( v_2 \). Then if we use an excitation \( i_1 + i_2 \), we will find a response \( v_1 + v_2 \).

The principle of superposition has the ability to reduce a complicated problem to several easier problems each containing only a single independent source.
Superposition theorem states that,

In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as algebraic sum of the individual contributions of each source acting alone.

When determining the contribution due to a particular independent source, we disable all the remaining independent sources. That is, all the remaining voltage sources are made zero by replacing them with short circuits, and all remaining current sources are made zero by replacing them with open circuits. Also, it is important to note that if a dependent source is present, it must remain active (unaltered) during the process of superposition.

**Action Plan:**

(i) In a circuit comprising of many independent sources, only one source is allowed to be active in the circuit, the rest are deactivated (turned off).

(ii) To deactivate a voltage source, replace it with a short circuit, and to deactivate a current source, replace it with an open circuit.

(iii) The response obtained by applying each source, one at a time, are then added algebraically to obtain a solution.

**Limitations:** Superposition is a fundamental property of linear equations and, therefore, can be applied to any effect that is linearly related to the cause. That is, we want to point out that, superposition principle applies only to the current and voltage in a linear circuit but it cannot be used to determine power because power is a non-linear function.

**EXAMPLE 3.1**

Find the current in the 6 Ω resistor using the principle of superposition for the circuit of Fig. 3.1.

![Circuit Diagram](image)

**SOLUTION**

As a first step, set the current source to zero. That is, the current source appears as an open circuit as shown in Fig. 3.2.

\[
 i_1 = \frac{6}{3+6} = \frac{6}{9} A 
\]

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As a next step, set the voltage to zero by replacing it with a short circuit as shown in Fig. 3.3.

\[
i_2 = \frac{2 \times 3}{3 + 6} = \frac{6}{9} \text{A}
\]

The total current \(i\) is then the sum of \(i_1\) and \(i_2\)

\[
i = i_1 + i_2 = \frac{12}{9} \text{A}
\]

**EXAMPLE 3.2**

Find \(i_\alpha\) in the network shown in Fig. 3.4 using superposition.

**SOLUTION**

As a first step, set the current source to zero. That is, the current source appears as an open circuit as shown in Fig. 3.5.
As a second step, set the voltage source to zero. This means the voltage source in Fig. 3.4 is replaced by a short circuit as shown in Figs. 3.6 and 3.6(a). Using current division principle,

\[ i_A = \frac{iR_2}{R_1 + R_2} \]

where

\[ R_1 = (12 \text{ kΩ} || 12 \text{ kΩ}) + 12 \text{ kΩ} = 6 \text{ kΩ} + 12 \text{ kΩ} = 18 \text{ kΩ} \]

and

\[ R_2 = 12 \text{ kΩ} \]

\[ i_A = \frac{4 \times 10^{-3} \times 12 \times 10^3}{(12 + 18) \times 10^3} = 1.6 \text{ mA} \]

Again applying the current division principle,

\[ i_o'' = \frac{i_A \times 12}{12 + 12} = 0.8 \text{ mA} \]

Thus,

\[ i_o = i_o' + i_o'' = -0.3 + 0.8 = 0.5 \text{ mA} \]
EXAMPLE 3.3

Use superposition to find $i_o$ in the circuit shown in Fig. 3.7.

Figure 3.7

SOLUTION

As a first step, keep only the 12 V source active and rest of the sources are deactivated. That is, 2 mA current source is opened and 6 V voltage source is shorted as shown in Fig. 3.8.

$$i'_o = \frac{12}{(2 + 2) \times 10^3} = 3 \text{ mA}$$

Figure 3.8

As a second step, keep only 6 V source active. Deactivate rest of the sources, resulting in a circuit diagram as shown in Fig. 3.9.
Applying KVL clockwise to the upper loop, we get

\[-2 \times 10^3 i_o'' - 2 \times 10^3 i_o''' - 6 = 0\]

\[\Rightarrow \quad i_o'' = \frac{-6}{4 \times 10^3} = -1.5 \text{ mA}\]

As a final step, deactivate all the independent voltage sources and keep only 2 mA current source active as shown in Fig. 3.10.

Current of 2 mA splits equally.

Hence,

\[i_o''' = 1 \text{ mA}\]

Applying the superposition principle, we find that

\[i_o = i_o' + i_o'' + i_o'''
\[= 3 - 1.5 + 1\]
\[= 2.5 \text{ mA}\]
EXAMPLE 3.4

Find the current $i$ for the circuit of Fig. 3.11.

![Figure 3.11](image)

SOLUTION

We need to find the current $i$ due to the two independent sources.

As a first step in the analysis, we will find the current resulting from the independent voltage source. The current source is deactivated and we have the circuit as shown as Fig. 3.12.

Applying KVL clockwise around loop shown in Fig. 3.12, we find that

$$5i_1 + 3i_1 - 24 = 0$$

$$\Rightarrow \quad i_1 = \frac{24}{8} = 3 \text{ A}$$

As a second step, we set the voltage source to zero and determine the current $i_2$ due to the current source. For this condition, refer to Fig. 3.13 for analysis.

![Figure 3.12](image) ![Figure 3.13](image)

Applying KCL at node 1, we get

$$i_2 + 7 = \frac{v_1 - 3i_2}{2} \quad (3.1)$$

Noting that

$$-i_2 = \frac{v_1 - 0}{3}$$

we get,

$$v_1 = -3i_2 \quad (3.2)$$
Making use of equation (3.2) in equation (3.1) leads to

\[ i_2 + 7 = \frac{-3i_2 - 3i_2}{2} \]

\[ \Rightarrow \quad i_2 = -\frac{7}{4} \text{ A} \]

Thus, the total current

\[ i = i_1 + i_2 = 3 - \frac{7}{4} \text{ A} = \frac{5}{4} \text{ A} \]

**EXAMPLE 3.5**

For the circuit shown in Fig. 3.14, find the terminal voltage \( V_{ab} \) using superposition principle.

As a first step in the analysis, deactivate the independent current source. This results in a circuit diagram as shown in Fig. 3.15.

*Applying KVL clockwise gives*

\[ -4 + 10 \times 0 + 3V_{cb_1} + V_{ab_1} = 0 \]

\[ \Rightarrow \quad 4V_{cb_1} = 4 \]

\[ \Rightarrow \quad V_{cb_1} = 1 \text{ V} \]

Next step in the analysis is to deactivate the independent voltage source, resulting in a circuit diagram as shown in Fig. 3.16.

*Applying KVL gives*

\[ -10 \times 2 + 3V_{cb_2} + V_{cb_2} = 0 \]

\[ \Rightarrow \quad 4V_{cb_2} = 20 \]

\[ \Rightarrow \quad V_{cb_2} = 5 \text{ V} \]
According to superposition principle,

\[ V_{ab} = V_{ab1} + V_{ab2} \\
= 1 + 5 = 6V \]

**EXAMPLE 3.6**

Use the principle of superposition to solve for \( v_x \) in the circuit of Fig. 3.17.

![Figure 3.17](image_url)

**SOLUTION**

According to the principle of superposition,

\[ v_x = v_{x1} + v_{x2} \]

where \( v_{x1} \) is produced by 6A source alone in the circuit and \( v_{x2} \) is produced solely by 4A current source.

To find \( v_{x1} \), deactivate the 4A current source. This results in a circuit diagram as shown in Fig. 3.18.

**KCL at node \( x_1 \):**

\[ \frac{v_{x1}}{2} + \frac{v_{x1} - 4i_{x1}}{8} = 6 \]

But

\[ i_{x1} = \frac{v_{x1}}{2} \]

Hence,

\[ \frac{v_{x1}}{2} + \frac{v_{x1} - 4(\frac{v_{x1}}{2})}{8} = 6 \]

\[ \Rightarrow \frac{v_{x1}}{2} + \frac{v_{x1} - 2v_{x1}}{8} = 6 \]

\[ \Rightarrow 4v_{x1} + v_{x1} - 2v_{x1} = 48 \]

\[ \Rightarrow v_{x1} = \frac{48}{3} = 16V \]
To find \( v_{x_2} \), deactivate the 6A current source, resulting in a circuit diagram as shown in Fig. 3.19.

**KCL at node \( x_2 \):**

\[
\frac{v_{x_2}}{8} + \frac{v_{x_2} - (-4i_{x_2})}{2} = 4
\]

\[
\Rightarrow \quad v_{x_2} + v_{x_2} + 4i_{x_2} = 4
\]

** Applying KVL along dotted path, we get **

\[
v_{x_2} + 4i_{x_2} - 2i_{x_2} = 0
\]

\[
\Rightarrow \quad v_{x_2} = -2i_{x_2} \quad \text{or} \quad i_{x_2} = -\frac{v_{x_2}}{2}
\]

Substituting equation (3.4) in equation (3.3), we get

\[
\frac{v_{x_2}}{8} + \frac{v_{x_2} + 4 \left(-\frac{v_{x_2}}{2}\right)}{2} = 4
\]

\[
\Rightarrow \quad \frac{v_{x_2}}{8} + \frac{v_{x_2} - 2v_{x_2}}{2} = 4
\]

\[
\Rightarrow \quad v_{x_2} - 4v_{x_2} = 32
\]

\[
\Rightarrow \quad v_{x_2} = -\frac{32}{3} \text{ V}
\]

Hence, according to the superposition principle,

\[
v_x = v_{x_1} + v_{x_2} = 16 - \frac{32}{2} = 5.33 \text{ V}
\]

**EXAMPLE 3.7**

Which of the source in Fig. 3.20 contributes most of the power dissipated in the 2 \( \Omega \) resistor? The least? What is the power dissipated in 2 \( \Omega \) resistor?
**SOLUTION**

The Superposition theorem cannot be used to identify the individual contribution of each source to the power dissipated in the resistor. However, the superposition theorem can be used to find the total power dissipated in the $2 \, \Omega$ resistor.

![Figure 3.21](image)

According to the superposition principle,

$$i_1 = i'_1 + i'_2$$

where $i'_1 = \text{Contribution to } i_1 \text{ from } 5V \text{ source alone.}$

and $i'_2 = \text{Contribution to } i_1 \text{ from } 2A \text{ source alone.}$

Let us first find $i'_1$. This needs the deactivation of $2A$ source. Refer to Fig. 3.22.

$$i'_1 = \frac{5}{2 + 2.1} = 1.22A$$

Similarly to find $i'_2$ we have to disable the $5V$ source by shorting it.

Referring to Fig. 3.23, we find that

$$i'_2 = \frac{-2 \times 2.1}{2 + 2.1} = -1.024 \, A$$

![Figure 3.22](image)

![Figure 3.23](image)
Total current,

\[ i_1 = i'_1 + i'_2 \]
\[ = 1.22 - 1.024 \]
\[ = 0.196 \text{ A} \]

Thus,

\[ P_{2\Omega} = (0.196)^2 \times 2 \]
\[ = 0.0768 \text{ Watts} \]
\[ = 76.8 \text{ mW} \]

**EXAMPLE 3.8**

Find the voltage \( V_1 \) using the superposition principle. Refer the circuit shown in Fig. 3.24.

![Figure 3.24](image)

**SOLUTION**

According to the superposition principle,

\[ V_1 = V'_1 + V''_1 \]

where \( V'_1 \) is the contribution from 60V source alone and \( V''_1 \) is the contribution from 4A current source alone.

To find \( V'_1 \), the 4A current source is opened, resulting in a circuit as shown in Fig. 3.25.

![Figure 3.25](image)
Applying KVL to the left mesh:

\[ 30i_a - 60 + 30 (i_a - i_b) = 0 \]  \hspace{1cm} (3.5)

Also

\[ i_b = -0.4i_A \]
\[ = -0.4 (-i_a) = 0.4i_a \] \hspace{1cm} (3.6)

Substituting equation (3.6) in equation (3.5), we get

\[ 30i_a - 60 + 30i_a - 30 \times 0.4i_a = 0 \]
\[ \Rightarrow \]
\[ i_a = \frac{60}{48} = 1.25 \text{A} \]
\[ i_b = 0.4i_a = 0.4 \times 1.25 \]
\[ = 0.5 \text{A} \]

Hence,

\[ V_1' = (i_a - i_b) \times 30 \]
\[ = 22.5 \text{V} \]

To find, \( V_1'' \), the 60V source is shorted as shown in Fig. 3.26.

![Figure 3.26](image)

Applying KCL at node a:

\[ \frac{V_a}{20} + \frac{V_a - V_1''}{10} = 4 \]
\[ \Rightarrow \]
\[ 30V_a - 20V_1'' = 800 \] \hspace{1cm} (3.7)

Applying KCL at node b:

\[ \frac{V_1''}{30} + \frac{V_1'' - V_a}{10} = 0.4i_b \]

Also,

\[ V_a = 20i_a \]
\[ \Rightarrow \]
\[ i_b = \frac{V_a}{20} \]

Hence,

\[ \frac{V_1''}{30} + \frac{V_1'' - V_a}{10} = 0.4V_a \]
\[ \Rightarrow \]
\[ -7.2V_a + 8V_1'' = 0 \] \hspace{1cm} (3.8)
Solving the equations (3.7) and (3.8), we find that

\[ V_1'' = 60V \]

Hence

\[ V_1 = V_1' + V_1'' = 22.5 + 60 = 82.5V \]

**EXAMPLE 3.9**

(a) Refer to the circuit shown in Fig. 3.27. Before the 10 mA current source is attached to terminals \( x - y \), the current \( i_a \) is found to be 1.5 mA. Use the superposition theorem to find the value of \( i_a \) after the current source is connected.

(b) Verify your solution by finding \( i_a \), when all the three sources are acting simultaneously.

![Figure 3.27](image)

**SOLUTION**

According to the principle of superposition,

\[ i_a = i_{a1} + i_{a2} + i_{a3} \]

where \( i_{a1}, i_{a2} \) and \( i_{a3} \) are the contributions to \( i_a \) from 20V source, 5 mA source and 10 mA source respectively.

As per the statement of the problem,

\[ i_{a1} + i_{a2} = 1.5 \text{ mA} \]

To find \( i_{a3} \), deactivate 20V source and the 5 mA source. The resulting circuit diagram is shown in Fig 3.28.

\[ i_{a3} = \frac{10\text{mA} \times 2k}{18k + 2k} = 1 \text{ mA} \]

Hence, total current

\[ i_a = i_{a1} + i_{a2} + i_{a3} = 1.5 + 1 = 2.5 \text{ mA} \]
(b) Refer to Fig. 3.29

**KCL at node y:**

\[
\frac{V_y}{18 \times 10^3} + \frac{V_y - 20}{2 \times 10^3} = (10+5) \times 10^{-3}
\]

Solving, we get \( V_y = 45\) V.

Hence, \( i_a = \frac{V_y}{18 \times 10^3} = \frac{45}{18 \times 10^3} = 2.5 \text{ mA} \)

### 3.2 Thevenin’s theorem

In section 3.1, we saw that the analysis of a circuit may be greatly reduced by the use of superposition principle. The main objective of Thevenin’s theorem is to reduce some portion of a circuit to an equivalent source and a single element. This reduced equivalent circuit connected to the remaining part of the circuit will allow us to find the desired current or voltage. Thevenin’s theorem is based on circuit equivalence. A circuit equivalent to another circuit exhibits identical characteristics at identical terminals.

According to Thevenin’s theorem, the linear circuit of Fig. 3.30 can be replaced by the one shown in Fig. 3.31 (The load resistor may be a single resistor or another circuit). The circuit to the left of the terminals \( x - y \) in Fig. 3.31 is known as the Thevenin’s equivalent circuit.
The Thevenin’s theorem may be stated as follows:

A linear two–terminal circuit can be replaced by an equivalent circuit consisting of a voltage source $V_t$ in series with a resistor $R_t$, Where $V_t$ is the open–circuit voltage at the terminals and $R_t$ is the input or equivalent resistance at the terminals when the independent sources are turned off or $R_t$ is the ratio of open–circuit voltage to the short–circuit current at the terminal pair.

Action plan for using Thevenin’s theorem:

1. Divide the original circuit into circuit $A$ and circuit $B$.

   ![Diagram](image1)

   In general, circuit $B$ is the load which may be linear or non-linear. Circuit $A$ is the balance of the original network exclusive of load and must be linear. In general, circuit $A$ may contain independent sources, dependent sources and resistors or other linear elements.

2. Separate the circuit $A$ from circuit $B$.

3. Replace circuit $A$ with its Thevenin’s equivalent.

4. Reconnect circuit $B$ and determine the variable of interest (e.g. current $i$ or voltage $v'$).

   ![Diagram](image2)

Procedure for finding $R_t$:

Three different types of circuits may be encountered in determining the resistance, $R_t$:

(i) If the circuit contains only independent sources and resistors, deactivate the sources and find $R_t$ by circuit reduction technique. Independent current sources, are deactivated by opening them while independent voltage sources are deactivated by shorting them.
(ii) If the circuit contains resistors, dependent and independent sources, follow the instructions described below:

(a) Determine the open circuit voltage $v_{oc}$ with the sources activated.
(b) Find the short circuit current $i_{sc}$ when a short circuit is applied to the terminals $a - b$.
(c) $R_t = \frac{v_{oc}}{i_{sc}}$

(iii) If the circuit contains resistors and only dependent sources, then

(a) $v_{oc} = 0$ (since there is no energy source)
(b) Connect 1A current source to terminals $a - b$ and determine $v_{ab}$.
(c) $R_t = \frac{v_{ab}}{1}$

For all the cases discussed above, the Thevenin’s equivalent circuit is as shown in Fig. 3.32.

**EXAMPLE 3.10**

Using the Thevenin’s theorem, find the current $i$ through $R = 2 \Omega$. Refer Fig. 3.33.

**SOLUTION**
Since we are interested in the current \( i \) through \( R \), the resistor \( R \) is identified as circuit B and the remainder as circuit A. After removing the circuit B, circuit A is as shown in Fig. 3.35.

![Figure 3.35](image)

To find \( R_t \), we have to deactivate the independent voltage source. Accordingly, we get the circuit in Fig. 3.36.

\[
R_t = (5 \, \Omega || 20 \, \Omega) + 4 \, \Omega = \frac{5 \times 20}{5 + 20} + 4 = 8 \, \Omega
\]

Referring to Fig. 3.35,

\[-50 + 25I = 0 \quad \Rightarrow \quad I = 2A\]

Hence \( V_{ab} = V_{oc} = 20(I) = 40V \)

Thus, we get the Thevenin’s equivalent circuit which is as shown in Fig.3.37.

![Figure 3.37](image) ![Figure 3.38](image)

Reconnecting the circuit B to the Thevenin’s equivalent circuit as shown in Fig. 3.38, we get

\[
\text{\( i = \frac{40}{2 + 8} = 4A \)}
\]
EXAMPLE 3.11
(a) Find the Thevenin’s equivalent circuit with respect to terminals \( a - b \) for the circuit shown in Fig. 3.39 by finding the open-circuit voltage and the short-circuit current.
(b) Solve the Thevenin resistance by removing the independent sources. Compare your result with the Thevenin resistance found in part (a).

Figure 3.39

SOLUTION
(a) To find \( V_{oc} \):
Apply KCL at node 2:
\[
\frac{V_2}{60 + 20} + \frac{V_2 - 30}{40} - 1.5 = 0
\]
\( \Rightarrow \)
\[
V_2 = 60 \text{ Volts}
\]
Hence,
\[
V_{oc} = I \times 60
\]
\[
= \left[ \frac{V_2 - 0}{60 + 20} \right] \times 60
\]
\[
= 60 \times \frac{60}{80} = 45 \text{ V}
\]
To find $i_{sc}$:

Applying KCL at node 2:

\[ \frac{V_2}{20} + \frac{V_2 - 30}{40} - 1.5 = 0 \]

\[ V_2 = 30\text{V} \]

\[ i_{sc} = \frac{V_2}{20} = 1.5\text{A} \]

Therefore,

\[ R_l = \frac{V_{oc}}{i_{sc}} = \frac{45}{1.5} = 30\text{Ω} \]

The Thevenin equivalent circuit with respect to the terminals $a - b$ is as shown in Fig. 3.40(a).

(b) Let us now find Thevenin resistance $R_t$ by deactivating all the independent sources.

\[ R_t = 60\text{Ω}|| (40 + 20)\text{Ω} \]

\[ = \frac{60}{2} = 30\text{Ω} \text{ (verified)} \]

It is seen that, if only independent sources are present, it is easy to find $R_t$ by deactivating all the independent sources.
EXAMPLE 3.12

Find the Thevenin equivalent for the circuit shown in Fig. 3.41 with respect to terminals $a - b$.

![Circuit Diagram](image)

SOLUTION

To find $V_{oc} = V_{ab}$:

Applying KVL around the mesh of Fig. 3.42, we get

\[-20 + 6i - 2i + 6i = 0\]

\[\Rightarrow \quad i = 2\text{A}\]

Since there is no current flowing in 10 $\Omega$ resistor, $V_{oc} = 6i = 12$ V

To find $R_t$: (Refer Fig. 3.43)

Since both dependent and independent sources are present, Thevenin resistance is found using the relation,

\[R_t = \frac{v_{oc}}{i_{sc}}\]

Applying KVL clockwise for mesh 1:

\[-20 + 6i_1 - 2i + 6(i_1 - i_2) = 0\]

\[\Rightarrow \quad 12i_1 - 6i_2 = 20 + 2i\]

Since $i = i_1 - i_2$, we get

\[12i_1 - 6i_2 = 20 + 2(i_1 - i_2)\]

\[\Rightarrow \quad 10i_1 - 4i_2 = 20\]

Applying KVL clockwise for mesh 2:

\[10i_2 + 6(i_2 - i_1) = 0\]

\[\Rightarrow \quad -6i_1 + 16i_2 = 0\]
Solving the above two mesh equations, we get
\[ i_2 = \frac{120}{136} \text{A} \quad \Rightarrow \quad i_{sc} = i_2 = \frac{120}{136} \text{A} \]
\[ R_t = \frac{v_{oc}}{i_{sc}} = \frac{12}{\frac{120}{136}} = 13.6 \Omega \]

**EXAMPLE 3.13**

Find \( V_o \) in the circuit of Fig. 3.44 using Thevenin’s theorem.

**Figure 3.44**

**SOLUTION**

To find \( V_{oc} \):

Since we are interested in the voltage across 2 \( k\Omega \) resistor, it is removed from the circuit of Fig. 3.44 and so the circuit becomes as shown in Fig. 3.45.

**Figure 3.45**

By inspection, \( i_1 = 4 \text{ mA} \)

**Applying KVL to mesh 2:**

\[-12 + 6 \times 10^3 (i_2 - i_1) + 3 \times 10^3 i_2 = 0 \]
\[\Rightarrow \quad -12 + 6 \times 10^3 (i_2 - 4 \times 10^{-3}) + 3 \times 10^3 i_2 = 0 \]
Solving, we get \( i_2 = 4 \text{ mA} \)

**Applying KVL to the path** \( 4 \text{k}\Omega \to a-b \to 3 \text{k}\Omega \), we get

\[
-4 \times 10^3 i_1 + V_{oc} - 3 \times 10^3 i_2 = 0
\]

\[
\Rightarrow V_{oc} = 4 \times 10^3 i_1 + 3 \times 10^3 i_2
\]

\[
= 4 \times 10^3 \times 4 \times 10^{-3} + 3 \times 10^3 \times 4 \times 10^{-3} = 28 \text{V}
\]

**To find** \( R_t \):

Deactivating all the independent sources, we get the circuit diagram shown in Fig. 3.46.

\[
R_t = R_{ab} = 4 \text{k}\Omega + (6 \text{k}\Omega || 3 \text{k}\Omega) = 6 \text{k}\Omega
\]

Hence, the Thevenin equivalent circuit is as shown in Fig. 3.47.

If we connect the \( 2 \text{k}\Omega \) resistor to this equivalent network, we obtain the circuit of Fig. 3.48.

\[
V_o = i \left(2 \times 10^3\right)
\]

\[
= \frac{28}{(6 + 2) \times 10^3} \times 2 \times 10^3 = 7 \text{V}
\]

**EXAMPLE 3.14**

The wheatstone bridge in the circuit shown in Fig. 3.49 (a) is balanced when \( R_2 = 1200 \Omega \). If the galvanometer has a resistance of \( 30 \Omega \), how much current will be detected by it when the bridge is unbalanced by setting \( R_2 \) to \( 1204 \Omega \)?
To find $V_{oc}$:

We are interested in the galvanometer current. Hence, it is removed from the circuit of Fig. 3.49 (a) to find $V_{oc}$ and we get the circuit shown in Fig. 3.49 (b).

$$i_1 = \frac{120}{900 + 600} = \frac{120}{1500} \text{A}$$

$$i_2 = \frac{120}{1204 + 800} = \frac{120}{2004} \text{A}$$

Applying KVL clockwise along the path $1204\Omega \rightarrow b - a \rightarrow 900 \Omega$, we get

$$1204i_2 - V_t - 900i_1 = 0$$

$$\Rightarrow V_t = 1204i_2 - 900i_1$$

$$= 1204 \times \frac{120}{2004} - 900 \times \frac{120}{1500}$$

$$= 95.8 \text{ mV}$$

To find $R_t$:

Deactivate all the independent sources and look into the terminals $a - b$ to determine the Thevenin’s resistance.
\[ R_t = R_{ab} = 600 | 900 + 800 | 1204 = \frac{900 \times 600}{1500} + \frac{1204 \times 800}{2004} = 840.64 \, \Omega \]

Hence, the Thevenin equivalent circuit consists of the 95.8 mV source in series with 840.64 \( \Omega \) resistor. If we connect 30\( \Omega \) resistor (galvanometer resistance) to this equivalent network, we obtain the circuit in Fig. 3.50.

\[ i_G = \frac{95.8 \times 10^{-3}}{840.64 + 30} = 110.03 \, \mu A \]

**EXAMPLE 3.15**

For the circuit shown in Fig. 3.51, find the Thevenin’s equivalent circuit between terminals \( a \) and \( b \).

**SOLUTION**

With \( ab \) shorted, let \( I_{sc} = I \). The circuit after transforming voltage sources into their equivalent current sources is as shown in Fig. 3.52.

Writing node equations for this circuit,

At \( a \) : \[ 0.2V_a - 0.1V_c + I = 3 \]

At \( c \) : \[ -0.1V_a + 0.3V_c - 0.1V_b = 4 \]

At \( b \) : \[ -0.1V_c + 0.2V_b - I = 1 \]

As the terminals \( a \) and \( b \) are shorted \( V_a = V_b \) and the above equations become
0.2V_a - 0.1 V_c + I = 3
-0.2V_a + 0.3 V_c = 4
0.2V_a - 0.1 V_c - 1 = 1

Solving the above equations, we get the short circuit current, \( I = I_{sc} = 1 \) A.

Next let us open circuit the terminals \( a \) and \( b \) and this makes \( I = 0 \). And the node equations written earlier are modified to

\[
0.2V_a - 0.1 V_c = 3 \\
-0.1V_a + 0.3 V_c - 0.1 V_b = 4 \\
-0.1V_c + 0.2 V_b = 1
\]

Solving the above equations, we get

\[ V_a = 30 \text{V} \text{ and } V_b = 20 \text{V} \]

Hence, \( V_{ab} = 30 - 20 = 10 \text{ V} = V_{oc} = V_t \)
Therefore \( R_t = \frac{V_{oc}}{I_{sc}} = \frac{10}{1} = 10 \Omega \)

The Thevenin’s equivalent is as shown in Fig 3.53

**Example 3.16**

Refer to the circuit shown in Fig. 3.54. Find the Thevenin equivalent circuit at the terminals \( a - b \).

To begin with let us transform 3 A current source and 10 V voltage source. This results in a network as shown in Fig. 3.55 (a) and further reduced to Fig. 3.55 (b).
Again transform the 30 V source and following the reduction procedure step by step from Fig. 3.55 (b) to 3.55 (d), we get the Thevenin’s equivalent circuit as shown in Fig. 3.56.

**EXAMPLE 3.17**

Find the Thevenin equivalent circuit as seen from the terminals $a - b$. Refer the circuit diagram shown in Fig. 3.57.
SOLUTION

Since the circuit has no independent sources, $i = 0$ when the terminals $a - b$ are open. Therefore, $V_{oc} = 0$.

The onus is now to find $R_t$. Since $V_{oc} = 0$ and $i_{sc} = 0$, $R_t$ cannot be determined from $R_t = \frac{V_{oc}}{i_{sc}}$. Hence, we choose to connect a source of 1 A at the terminals $a - b$ as shown in Fig. 3.58. Then, after finding $V_{ab}$, the Thevenin resistance is,

$$R_t = \frac{V_{ab}}{1}$$

**KCL at node a**:

$$\frac{V_a - 2i}{5} + \frac{V_a}{10} - 1 = 0$$

Also,

$$i = \frac{V_a}{10}$$

Hence,

$$\frac{V_a - 2\left(\frac{V_a}{10}\right)}{5} + \frac{V_a}{10} - 1 = 0$$

$$\Rightarrow \quad V_a = \frac{50}{13} \text{V}$$

Hence,

$$R_t = \frac{V_a}{1} = \frac{50}{13} \Omega$$

Alternatively one could find $R_t$ by connecting a 1V source at the terminals $a - b$ and then find the current from $b$ to $a$. Then $R_t = \frac{1}{i_{ba}}$. The concept of finding $R_t$ by connecting a 1A source between the terminals $a - b$ may also be used for circuits containing independent sources. Then set all the independent sources to zero and use 1A source at the terminals $a - b$ to find $V_{ab}$ and hence, $R_t = \frac{V_{ab}}{1}$.

For the present problem, the Thevenin equivalent circuit as seen between the terminals $a - b$ is shown in Fig. 3.58 (a).
EXAMPLE 3.18

Determine the Thevenin equivalent circuit between the terminals \( a - b \) for the circuit of Fig. 3.59.

![Figure 3.59](image)

\[ \frac{1}{2} V_x \]
\[ + \]
\[ \frac{1}{2} V_x \]
\[ 4\Omega \]
\[ V_x \]
\[ - \]
\[ b \]

**SOLUTION**

As there are no independent sources in the circuit, we get \( V_{oc} = V_I = 0 \).

To find \( R_t \), connect a 1V source to the terminals \( a - b \) and measure the current \( I \) that flows from \( b \) to \( a \). (Refer Fig. 3.60 a).

\[ R_t = \frac{1}{I} \Omega \]

![Figure 3.60(a)](image)

**Applying KCL at node \( a \):**

\[ I = 0.5V_x + \frac{V_x}{4} \]

Since, \( V_x = 1V \)

we get,

\[ I = 0.5 + \frac{1}{4} = 0.75 \text{ A} \]

Hence,

\[ R_t = \frac{1}{0.75} = 1.33 \Omega \]

The Thevenin equivalent circuit is shown in 3.60(b).

Alternatively, sticking to our strategy, let us connect 1A current source between the terminals \( a - b \) and then measure \( V_{ab} \) (Fig. 3.60 (c)). Consequently, \( R_t = \frac{V_{ab}}{I} = V_{ab} \Omega \).
Applying KCL at node $a$:

$$0.5V_x + \frac{V_x}{4} = 1 \Rightarrow V_x = 1.33V$$

Hence

$$R_t = \frac{V_{ab}}{I} = \frac{V_x}{1} = 1.33 \, \Omega$$

The corresponding Thevenin equivalent circuit is same as shown in Fig. 3.60(b).

### 3.3 Norton’s theorem

An American engineer, E.L. Norton at Bell Telephone Laboratories, proposed a theorem similar to Thevenin’s theorem.

**Norton’s theorem states that a linear two-terminal network can be replaced by an equivalent circuit consisting of a current source $i_N$ in parallel with resistor $R_N$, where $i_N$ is the short-circuit current through the terminals and $R_N$ is the input or equivalent resistance at the terminals when the independent sources are turned off. If one does not wish to turn off the independent sources, then $R_N$ is the ratio of open circuit voltage to short-circuit current at the terminal pair.**

Figure 3.61(b) shows Norton’s equivalent circuit as seen from the terminals $a - b$ of the original circuit shown in Fig. 3.61(a). Since this is the dual of the Thevenin circuit, it is clear that $R_N = R_t$ and $i_N = \frac{V_{oc}}{R_t}$. In fact, source transformation of Thevenin equivalent circuit leads to Norton’s equivalent circuit.

**Procedure for finding Norton’s equivalent circuit:**

1. If the network contains resistors and independent sources, follow the instructions below:
   
   (a) Deactivate the sources and find $R_N$ by circuit reduction techniques.
   
   (b) Find $i_N$ with sources activated.

2. If the network contains resistors, independent and dependent sources, follow the steps given below:

   (a) Determine the short-circuit current $i_N$ with all sources activated.
(b) Find the open-circuit voltage \( v_{oc} \).

(c) \( R_t = R_N = \frac{v_{oc}}{i_N} \)

3) If the network contains only resistors and dependent sources, follow the procedure described below:

(a) Note that \( i_N = 0 \).

(b) Connect 1A current source to the terminals \( a-b \) and find \( v_{ab} \).

(c) \( R_t = \frac{v_{ab}}{I} \)

Note: Also, since \( v_t = v_{oc} \) and \( i_N = i_{sc} \)

\[ R_t = \frac{v_{oc}}{i_{sc}} = R_N \]

The open–circuit and short–circuit test are sufficient to find any Thevenin or Norton equivalent.

### 3.3.1 PROOF OF THEVENIN’S AND NORTON’S THEOREMS

The principle of superposition is employed to provide the proof of Thevenin’s and Norton’s theorems.

**Derivation of Thevenin’s theorem:**

Let us consider a linear circuit having two accessible terminals \( x - y \) and excited by an external current source \( i \). The linear circuit is made up of resistors, dependent and independent sources. For the sake of simplified analysis, let us assume that the linear circuit contains only two independent voltage sources \( v_1 \) and \( v_2 \) and two independent current sources \( i_1 \) and \( i_2 \). The terminal voltage \( v \) may be obtained, by applying the principle of superposition. That is, \( v \) is made up of contributions due to the external source and independent sources within the linear network.

Hence,

\[ v = a_0i + a_1v_1 + a_2v_2 + a_3i_1 + a_4i_2 \]  \hspace{1cm} (3.9)

\[ = a_0i + b_0 \]  \hspace{1cm} (3.10)

where

\[ b_0 = a_1v_1 + a_2v_2 + a_3i_1 + a_4i_2 \]

= contribution to the terminal voltage \( v \) by independent sources within the linear network.

Let us now evaluate the values of constants \( a_0 \) and \( b_0 \).

(i) When the terminals \( x \) and \( y \) are open–circuited, \( i = 0 \) and \( v = v_{oc} = v_t \). Making use of this fact in equation 3.10, we find that \( b_0 = v_t \).
(ii) When all the internal sources are deactivated, \( b_0 = 0 \). This enforces equation 3.10 to become

\[
v = a_0 i = R_e i \Rightarrow a_0 = R_e
\]

where \( R_e \) is the equivalent resistance of the linear network as viewed from the terminals \( x - y \). Also, \( a_0 \) must be \( R_e \) in order to obey the ohm’s law. Substituting the values of \( a_0 \) and \( b_0 \) in equation 3.10, we find that

\[
v = R_e i + v_1
\]

which expresses the voltage-current relationship at terminals \( x - y \) of the circuit in Fig. 3.63. Thus, the two circuits of Fig. 3.62 and 3.63 are equivalent.

**Derivation of Norton’s theorem:**

Let us now assume that the linear circuit described earlier is driven by a voltage source \( v \) as shown in Fig. 3.64.

The current flowing into the circuit can be obtained by superposition as

\[
i = c_0 v + d_0 \quad (3.11)
\]

where \( c_0 v \) is the contribution to \( i \) due to the external voltage source \( v \) and \( d_0 \) contains the contributions to \( i \) due to all independent sources within the linear circuit. The constants \( c_0 \) and \( d_0 \) are determined as follows:

(i) When terminals \( x - y \) are short-circuited, \( v = 0 \) and \( i = -i_{sc} \). Hence from equation (3.11), we find that \( i = d_0 = -i_{sc} \), where \( i_{sc} \) is the short-circuit current flowing out of terminal \( x \), which is same as Norton current \( i_N \)

Thus,

\[
d_0 = -i_N
\]

(ii) Let all the independent sources within the linear network be turned off, that is \( d_0 = 0 \). Then, equation (3.11) becomes

\[
i = c_0 v
\]
For dimensional validity, $c_0$ must have the dimension of conductance. This enforces $c_0 = \frac{1}{R_t}$ where $R_t$ is the equivalent resistance of the linear network as seen from the terminals $x - y$. Thus, equation (3.11) becomes

$$i = \frac{1}{R_t}v - i_{sc}$$

$$= \frac{1}{R_t}v - i_N$$

This expresses the voltage-current relationship at the terminals $x - y$ of the circuit in Fig. (3.65), validating that the two circuits of Figs. 3.64 and 3.65 are equivalents.

**EXAMPLE 3.19**

Find the Norton equivalent for the circuit of Fig. 3.66.

![Figure 3.66](image_url)

**SOLUTION**

As a first step, short the terminals $a - b$. This results in a circuit diagram as shown in Fig. 3.67. Applying KCL at node $a$, we get

$$\frac{0 - 24}{4} - 3 + i_{sc} = 0$$

$$\Rightarrow i_{sc} = 9\text{ A}$$

To find $R_N$, deactivate all the independent sources, resulting in a circuit diagram as shown in Fig. 3.68 (a). We find $R_N$ in the same way as $R_t$ in the Thevenin equivalent circuit.

$$R_N = \frac{4 \times 12}{4 + 12} = 3\ \Omega$$

![Figure 3.67](image_url)
Thus, we obtain Norton equivalent circuit as shown in Fig. 3.68(b).

**EXAMPLE 3.20**
Refer the circuit shown in Fig. 3.69. Find the value of $i_b$, using Norton equivalent circuit. Take $R = 667 \, \Omega$.

![Figure 3.69](image)

**SOLUTION**
Since we want the current flowing through $R$, remove $R$ from the circuit of Fig. 3.69. The resulting circuit diagram is shown in Fig. 3.70.

To find $i_{ac}$ or $i_N$ referring Fig 3.70(a):

$$i_a = \frac{0}{1000} = 0 \, \text{A}$$

$$i_{sc} = \frac{12}{6000} = 2 \, \text{mA}$$
To find $R_N$:
The procedure for finding $R_N$ is same that of $R_t$ in the Thevenin equivalent circuit.

$$R_t = R_N = \frac{v_{oc}}{i_{sc}}$$

To find $v_{oc}$, make use of the circuit diagram shown in Fig. 3.71. Do not deactivate any source.

*Applying KVL clockwise,* we get

$$-12 + 6000i_a + 2000i_a + 1000i_a = 0$$

$$\Rightarrow i_a = \frac{4}{3000} \text{ A}$$

$$\Rightarrow v_{oc} = i_a \times 1000 = \frac{4}{3} \text{ V}$$

Therefore,

$$R_N = \frac{v_{oc}}{i_{sc}} = \frac{4}{2 \times 10^{-3}} = 667 \Omega$$

The Norton equivalent circuit along with resistor $R$ is as shown below:

$$i_b = \frac{i_{sc}}{2} = \frac{2 \text{ mA}}{2} = 1 \text{ mA}$$

**EXAMPLE 3.21**

Find $I_o$ in the network of Fig. 3.72 using Norton’s theorem.
**SOLUTION**

We are interested in $I_o$, hence the $2 \, \text{k}\Omega$ resistor is removed from the circuit diagram of Fig. 3.72. The resulting circuit diagram is shown in Fig. 3.73(a).

![Figure 3.73(a)](image1)

**To find $i_N$ or $i_{sc}$:**

Refer Fig. 3.73(b). By inspection, $V_1 = 12 \, \text{V}$

Applying KCL at node $V_2$:

$$\frac{V_2 - V_1}{6 \, \text{k}\Omega} + \frac{V_2}{2 \, \text{k}\Omega} + \frac{V_2 - V_1}{3 \, \text{k}\Omega} = 0$$

Substituting $V_1 = 12 \, \text{V}$ and solving, we get

$$V_2 = 6 \, \text{V}$$

$$i_{sc} = \frac{V_1 - V_2}{3 \, \text{k}\Omega} + \frac{V_1}{4 \, \text{k}\Omega} = 5 \, \text{mA}$$

**To find $R_N$:**

Deactivate all the independent sources (refer Fig. 3.73(c)).

![Figure 3.73(c)](image2)
Referring to Fig. 3.73 (d), we get

\[ R_N = R_{ab} = 4 \, \text{k} \Omega \parallel [3 \, \text{k} \Omega + (6 \, \text{k} \Omega \parallel 2 \, \text{k} \Omega)] = 2.12 \, \text{k} \Omega \]

Hence, the Norton equivalent circuit along with 2 \, \text{k} \Omega resistor is as shown in Fig. 3.73(e).

\[ I_o = \frac{i_{sc} \times R_N}{R + R_N} = \frac{5 \, \text{mA} \times 2.12 \, \text{k} \Omega}{2 \, \text{k} \Omega + 2.12 \, \text{k} \Omega} = 2.57 \, \text{mA} \]

**EXAMPLE 3.22**

Find \( V_o \) in the circuit of Fig. 3.74.

**SOLUTION**

Since we are interested in \( V_o \), the voltage across 4 \, \text{k} \Omega resistor, remove this resistance from the circuit. This results in a circuit diagram as shown in Fig. 3.75.
To find $i_{ac}$, short the terminals $a - b$:

\[ \frac{6}{3k} = 2mA \]

\[ 2mA \]

\[ 2mA \times 2k\Omega = 4V \]
Constraint equation:

\[ i_1 - i_2 = 4 \text{mA} \]  
(3.12)

KVL around supermesh:

\[ -4 + 2 \times 10^3 i_1 + 4 \times 10^3 i_2 = 0 \]  
(3.13)

KVL around mesh 3:

\[ 8 \times 10^3(i_3 - i_2) + 2 \times 10^3(i_3 - i_1) = 0 \]

Since \( i_3 = i_{sc} \), the above equation becomes,

\[ 8 \times 10^3(i_{sc} - i_2) + 2 \times 10^3(i_{sc} - i_1) = 0 \]  
(3.14)

Solving equations (3.12), (3.13) and (3.14) simultaneously, we get \( i_{sc} = 0.1333 \text{ mA} \).

To find \( R_N \): 

Deactivate all the sources in Fig. 3.75. This yields a circuit diagram as shown in Fig. 3.76.

![Circuit Diagram](image)

\[ R_N = 6 \text{ k}\Omega || 10 \text{ k}\Omega \]
\[ = \frac{6 \times 10}{6 + 10} = 3.75 \text{ k}\Omega \]

Hence, the Norton equivalent circuit is as shown in Fig 3.76 (a).

To the Norton equivalent circuit, now connect the 4 k\( \Omega \) resistor that was removed earlier to get the network shown in Fig. 3.76(b).
EXAMPLE 3.23

Find the Norton equivalent to the left of the terminals $a - b$ for the circuit of Fig. 3.77.

\[ V_o = i_{sc} (R_N || R) \]
\[ = i_{sc} \frac{R_N R}{R_N + R} \]
\[ = 258 \text{ mV} \]

Figure 3.76(b) Norton equivalent circuit with $R = 4 \text{ k}\Omega$

SOLUTION

To find $i_{sc}$:

Note that $v_{ab} = 0$ when the terminals $a - b$ are short-circuited.

Then \[ i = \frac{5}{500} = 10 \text{ mA} \]

Therefore, for the right-hand portion of the circuit, $i_{sc} = -10i = -100 \text{ mA}$. 
To find $R_N$ or $R_t$:

Writing the KVL equations for the left-hand mesh, we get

$$-5 + 500i + v_{ab} = 0 \quad (3.15)$$

Also for the right-hand mesh, we get

$$v_{ab} = -25(10i) = -250i$$

Therefore

$$i = \frac{-v_{ab}}{250}$$

Substituting $i$ into the mesh equation (3.15), we get

$$-5 + 500 \left( \frac{-v_{ab}}{250} \right) + v_{ab} = 0$$

$$\Rightarrow R_N = R_t \triangleq \frac{v_{oc}}{i_{sc}} = \frac{v_{ab}}{i_{sc}} = \frac{-5}{-0.1} = 50 \ \Omega$$

The Norton equivalent circuit is shown in Fig 3.77 (a).

Example 3.24

Find the Norton equivalent of the network shown in Fig. 3.78.
Since there are no independent sources present in the network of Fig. 3.78, \( i_N = i_{nc} = 0 \).

To find \( R_N \), we inject a current of 1A between the terminals \( a - b \). This is illustrated in Fig. 3.79.

![Figure 3.79](image)

**KCL at node 1:**

\[
1 = \frac{v_1}{100} + \frac{v_1 - v_2}{50} = 0.03v_1 - 0.02v_2 = 1
\]

**KCL at node 2:**

\[
\frac{v_2}{200} + \frac{v_2 - v_1}{50} + 0.1v_1 = 0
\]

\[
\frac{0.08v_1 + 0.025v_2}{200} = 0
\]

Solving the above two nodal equations, we get

\[
v_1 = 10.64\ \text{volts} \quad \Rightarrow \quad v_{oc} = 10.64\ \text{volts}
\]

Hence,

\[
R_N = R_e = \frac{v_{oc}}{\frac{1}{1}} = \frac{10.64}{1} = 10.64\ \Omega
\]

Norton equivalent circuit for the network shown in Fig. 3.78 is as shown in Fig. 3.79(a).

**EXAMPLE 3.25**

Find the Thevenin and Norton equivalent circuits for the network shown in Fig. 3.80 (a).

![Figure 3.80(a)](image)
SOLUTION

To find $V_{oc}$:

Performing source transformation on 5A current source, we get the circuit shown in Fig. 3.80 (b).

Applying KVL around Left mesh:

$$-50 + 2i_\alpha - 20 + 4i_\alpha = 0$$

$$\Rightarrow \quad i_\alpha = \frac{70}{6} \text{A}$$

Applying KVL around right mesh:

$$20 + 10i_\alpha + V_{oc} - 4i_\alpha = 0$$

$$\Rightarrow \quad V_{oc} = -90 \text{V}$$

To find $i_{sc}$ (referring Fig 3.80 (c)):

**KVL around Left mesh**:

$$-50 + 2i_\alpha - 20 + 4(i_\alpha - i_{sc}) = 0$$

$$\Rightarrow \quad 6i_\alpha - 4i_{sc} = 70$$

**KVL around right mesh**:

$$4(i_{sc} - i_\alpha) + 20 + 10i_\alpha = 0$$

$$\Rightarrow \quad 6i_\alpha + 4i_{sc} = -20$$

Solving the two mesh equations simultaneously, we get $i_{sc} = -11.25 \text{A}$

Hence, $R_t = R_N = \frac{v_{oc}}{i_{sc}} = \frac{-90}{-11.25} = 8 \Omega$

Performing source transformation on Thevenin equivalent circuit, we get the norton equivalent circuit (both are shown below).
EXAMPLE 3.26

If an 8 kΩ load is connected to the terminals of the network in Fig. 3.81, \( V_{AB} = 16 \text{ V} \). If a 2 kΩ load is connected to the terminals, \( V_{AB} = 8 \text{ V} \). Find \( V_{AB} \) if a 20 kΩ load is connected across the terminals.

**SOLUTION**

Applying KVL around the mesh, we get \((R_t + R_L)I = V_{oc}\)

If \( R_L = 2 \text{ kΩ} \), \( I = 10 \text{ mA} \) \( \Rightarrow \) \( V_{oc} = 20 + 0.01R_t \)

If \( R_L = 10 \text{ kΩ} \), \( I = 6 \text{ mA} \) \( \Rightarrow \) \( V_{oc} = 60 + 0.006R_t \)

Solving, we get \( V_{oc} = 120 \text{ V}, R_t = 10 \text{ kΩ} \).

If \( R_L = 20 \text{ kΩ} \), \( I = \frac{V_{oc}}{(R_L + R_t)} = \frac{120}{(20 \times 10^3 + 10 \times 10^3)} = 4 \text{ mA} \)

### 3.4 Maximum Power Transfer Theorem

In circuit analysis, we are sometimes interested in determining the maximum power that a circuit can supply to the load. Consider the linear circuit A as shown in Fig. 3.82.

Circuit A is replaced by its Thevenin equivalent circuit as seen from \( a \) and \( b \) (Fig 3.83).

We wish to find the value of the load \( R_L \) such that the maximum power is delivered to it.

The power that is delivered to the load is given by

\[
p = i^2R_L = \left[ \frac{V_t}{R_t + R_L} \right]^2 R_L
\]  

(3.16)
Assuming that $V_t$ and $R_t$ are fixed for a given source, the maximum power is a function of $R_L$. In order to determine the value of $R_L$ that maximizes $p$, we differentiate $p$ with respect to $R_L$ and equate the derivative to zero.

$$\frac{dp}{dR_L} = V_t^2 \left[ \frac{(R_t + R_L)^2 - 2(R_t + R_L)}{(R_L + R_t)^2} \right] = 0$$

which yields

$$R_L = R_t$$

To confirm that equation (3.17) is a maximum, it should be shown that $\frac{d^2p}{dR_L^2} < 0$. Hence, maximum power is transferred to the load when $R_L$ is equal to the Thevenin equivalent resistance $R_t$.

The maximum power transferred to the load is obtained by substituting $R_L = R_t$ in equation 3.16.

Accordingly,

$$P_{\text{max}} = \frac{V_t^2 R_t}{(2R_t)^2} = \frac{V_t^2}{4R_t}$$

The maximum power transfer theorem states that the maximum power delivered by a source represented by its Thevenin equivalent circuit is attained when the load $R_L$ is equal to the Thevenin resistance $R_t$.

**EXAMPLE 3.27**

Find the load $R_L$ that will result in maximum power delivered to the load for the circuit of Fig. 3.84. Also determine the maximum power $P_{\text{max}}$.

**SOLUTION**

Disconnect the load resistor $R_L$. This results in a circuit diagram as shown in Fig. 3.85(a).

Next let us determine the Thevenin equivalent circuit as seen from $a - b$. 

![Figure 3.83 Thevenin equivalent circuit](image-url)


\[ i = \frac{180}{150 + 30} = 1 \text{A} \]

\[ V_{oc} = V_t = 150 \times i = 150 \text{V} \]

To find \( R_t \), deactivate the 180 V source. This results in the circuit diagram of Fig. 3.85(b).

\[ R_t = R_{ab} = 30 \text{Ω} \mid 150 \text{Ω} \]

\[ = \frac{30 \times 150}{30 + 150} = 25 \text{Ω} \]

The Thevenin equivalent circuit connected to the load resistor is shown in Fig. 3.86.

Maximum power transfer is obtained when \( R_L = R_t = 25 \text{Ω} \).

Then the maximum power is

\[ P_{max} = \frac{V_t^2}{4R_L} = \frac{(150)^2}{4 \times 25} = 2.25 \text{Watts} \]

The Thevenin source \( V_t \) actually provides a total power of

\[ P_t = 150 \times i \]

\[ = 150 \times \left(\frac{150}{25 + 25}\right) \]

\[ = 450 \text{Watts} \]

Thus, we note that one-half the power is dissipated in \( R_L \).

**EXAMPLE 3.28**

Refer to the circuit shown in Fig. 3.87. Find the value of \( R_L \) for maximum power transfer. Also find the maximum power transferred to \( R_L \).
SOLUTION
Disconnecting $R_L$, results in a circuit diagram as shown in Fig. 3.88(a).

![Figure 3.88(a)](image)

To find $R_t$, deactivate all the independent voltage sources as in Fig. 3.88(b).

![Figure 3.88(b)](image)

$$R_t = R_{ab} = 6 \, \text{k}\Omega || 6 \, \text{k}\Omega || 6 \, \text{k}\Omega = 2 \, \text{k}\Omega$$

To find $V_t$:
Refer the Fig. 3.88(d).
Constraint equation:
$$V_3 - V_1 = 12 \, \text{V}$$

By inspection,
$$V_2 = 3 \, \text{V}$$

KCL at supernode:
$$\frac{V_3 - V_2}{6k} + \frac{V_1}{6k} + \frac{V_1 - V_2}{6k} = 0$$

$$\Rightarrow \quad \frac{V_3 - 3}{6k} + \frac{V_3 - 12}{6k} + \frac{V_3 - 12 - 3}{6k} = 0$$
The Thevenin equivalent circuit connected to the load resistor $R_L$ is shown in Fig. 3.88(e).

$$P_{\text{max}} = i^2 R_L = \left[ \frac{V_t}{2 R_L} \right]^2 R_L = 12.5 \text{ mW}$$

**Alternate method**:

It is possible to find $P_{\text{max}}$, without finding the Thevenin equivalent circuit. However, we have to find $R_t$. For maximum power transfer, $R_L = R_t = 2 \, k\Omega$. Insert the value of $R_L$ in the original circuit given in Fig. 3.87. Then use any circuit reduction technique of your choice to find power dissipated in $R_L$.

Refer Fig. 3.88(f). By inspection we find that, $V_2 = 3 \, V$.

**Constraint equation**:

$$V_3 - V_1 = 12$$

$$\Rightarrow V_1 = V_3 - 12$$

**KCL at supernode**:

$$\frac{V_3 - V_2}{6k} + \frac{V_1 - V_2}{6k} + \frac{V_3 + V_1}{2k} = 0$$

$$\Rightarrow \frac{V_3 - 3}{6k} + \frac{V_3 - 12 - 3}{6k} + \frac{V_3 - 12}{2k} = 0$$

$$\Rightarrow V_3 - 3 + V_3 - 15 + 3V_3 + V_3 - 12 = 0$$

$$\Rightarrow 6V_3 = 30$$

$$\Rightarrow V_3 = 5 \, V$$

Hence,

$$P_{\text{max}} = \frac{V_3^2}{R_L} = \frac{25}{2k} = 12.5 \text{ mW}$$
EXAMPLE 3.29

Find $R_L$ for maximum power transfer and the maximum power that can be transferred in the network shown in Fig. 3.89.

![Figure 3.89](image)

SOLUTION

Disconnect the load resistor $R_L$. This results in a circuit as shown in Fig. 3.89(a).

![Figure 3.89(a)](image)

To find $R_t$, let us deactivate all the independent sources, which results the circuit as shown in Fig. 3.89(b).

$$R_t = R_{eq} = 2 \, \text{k}\Omega + 3 \, \text{k}\Omega + 5 \, \text{k}\Omega = 10 \, \text{k}\Omega$$

For maximum power transfer $R_{tL} = R_t = 10 \, \text{k}\Omega$.

Let us next find $V_{oc}$ or $V_t$.

Refer Fig. 3.89 (c). By inspection, $i_1 = -2 \, \text{mA}$ & $i_2 = 1 \, \text{mA}$. 
Applying KVL clockwise to the loop $5 \, k\Omega \rightarrow 3 \, k\Omega \rightarrow 2 \, k\Omega \rightarrow a - b$, we get

$$-5k \times i_2 + 3k \left( i_1 - i_2 \right) + 2k \times i_1 + V_t = 0$$

$$\Rightarrow -5 \times 10^3 (1 \times 10^{-3}) + 3 \times 10^3 \left( -2 \times 10^{-3} - 1 \times 10^{-3} \right) + 2 \times 10^3 \left( -2 \times 10^{-3} \right) + V_t = 0$$

$$\Rightarrow -5 - 9 - 4 + V_t = 0$$

$$\Rightarrow V_t = 18 \, V.$$

The Thevenin equivalent circuit with load resistor $R_L$ is as shown in Fig. 3.89 (d).

Then,

$$i = \frac{18}{(10 + 10) \times 10^3} = 0.9 \, mA$$

Then,

$$P_{max} = P_L = (0.9 \, mA)^2 \times 10 \, k\Omega$$

$$= 8.1 \, mW$$

**EXAMPLE 3.30**

Find the maximum power dissipated in $R_L$. Refer the circuit shown in Fig. 3.90.
**SOLUTION**

Disconnecting the load resistor $R_L$ from the original circuit results in a circuit diagram as shown in Fig. 3.91.

![Circuit Diagram](fig3.91)

As a first step in the analysis, let us find $R_L$. While finding $R_L$, we have to deactivate all the independent sources. This results in a network as shown in Fig 3.91 (a):

![Network Diagram](fig3.91a)

$$R_L = R_{eq} = \frac{140 \Omega |60 \Omega | 8 \Omega}{140 + 60} = 50 \Omega.$$

For maximum power transfer, $R_L = R_t = 50 \Omega$. Next step in the analysis is to find $V_t$.

Refer Fig 3.91(b), using the principle of current division,

$$i_1 = \frac{i \times R_2}{R_1 + R_2} = \frac{20 \times 170}{170 + 30} = 17 \text{ A}$$

$$i_2 = \frac{i \times R_1}{R_1 + R_2} = \frac{20 \times 30}{170 + 30} = \frac{600}{200} = 3 \text{ A}$$

![Current Division Diagram](fig3.91a)
Applying KVL clockwise to the loop comprising of 50 Ω → 10 Ω → 8 Ω → a → b, we get

\[ 50i_2 - 10i_1 + 8 \times 0 + V_t = 0 \]
\[ \Rightarrow 50(3) - 10(17) + V_t = 0 \]
\[ \Rightarrow V_t = 20 \text{ V} \]

The Thevenin equivalent circuit with load resistor \( R_L \) is as shown in Fig. 3.91(c).

\[ i_T = \frac{20}{50 + 50} = 0.2 \text{ A} \]
\[ P_{\text{max}} = i_T^2 \times 50 = 0.04 \times 50 = 2 \text{ W} \]

**EXAMPLE 3.31**

Find the value of \( R_L \) for maximum power transfer in the circuit shown in Fig. 3.92. Also find \( P_{\text{max}} \).

**SOLUTION**

Disconnecting \( R_L \) from the original circuit, we get the network shown in Fig. 3.93.
Let us draw the Thevenin equivalent circuit as seen from the terminals \(a - b\) and then insert the value of \(R_L = R_t\) between the terminals \(a - b\). To find \(R_t\), let us deactivate all independent sources which results in the circuit as shown in Fig. 3.94.

![Figure 3.94](image)

\[
R_t = R_{ab} = 8 \, \Omega \parallel 2 \, \Omega = \frac{8 \times 2}{8 + 2} = 1.6 \, \Omega
\]

Next step is to find \(V_{oc}\) or \(V_t\).

By performing source transformation on the circuit shown in Fig. 3.93, we obtain the circuit shown in Fig. 3.95.

![Figure 3.95](image)

**Applying KVL to the loop** made up of 20 V → 3 Ω → 2 Ω → 10 V → 5 Ω → 30 V, we get

\[
-20 + 10i - 10 - 30 = 0
\]

\[
\Rightarrow \quad i = \frac{60}{10} = 6 \, A
\]
Again applying KVL clockwise to the path \( 2 \Omega \rightarrow 10 \text{ V} \rightarrow a \rightarrow b \), we get

\[
2i - 10 - V_t = 0
\]

\[
\Rightarrow V_t = 2i - 10
\]

\[
= 2(6) - 10 = 2 \text{ V}
\]

The Thevenin equivalent circuit with load resistor \( R_L \) is as shown in Fig. 3.95 (a).

\[
P_{\text{max}} = i_T^2 R_L
\]

\[
= \frac{V_t^2}{4R_t} = 625 \text{ mW}
\]

**EXAMPLE 3.32**

Find the value of \( R_L \) for maximum power transfer. Hence find \( P_{\text{max}} \).

**SOLUTION**

Removing \( R_L \) from the original circuit gives us the circuit diagram shown in Fig. 3.97.

\[
\text{To find } V_{oc}:
\]

**KCL at node } A :**

\[
-i'_a - 0.9 + 10i'_a = 0
\]

\[
\Rightarrow i'_a = 0.1 \text{ A}
\]

Hence,

\[
V_{oc} = 3 (10i'_a)
\]

\[
= 3 \times 10 \times 0.1 = 3 \text{ V}
\]
To find $R_L$, we need to compute $i_{sc}$ with all independent sources activated.

**KCL at node A:**

$$-i_a'' - 0.9 + 10i_a'' = 0$$

$$\Rightarrow i_a'' = 0.1 \text{ A}$$

Hence $i_{sc} = 10i_a'' = 10 \times 0.1 = 1 \text{ A}$

$$R_t = \frac{V_{ac}}{i_{sc}} = \frac{3}{1} = 3 \text{ \Omega}$$

Hence, for maximum power transfer $R_L = R_t = 3 \text{ \Omega}$.

The Thevenin equivalent circuit with $R_L = 3 \text{ \Omega}$ inserted between the terminals $a - b$ gives the network shown in Fig. 3.97(a).

$$i_T = \frac{3}{3 + 3} = 0.5 \text{ A}$$

$$P_{\text{max}} = i_T^2 R_L$$

$$= (0.5)^2 \times 3$$

$$= 0.75 \text{ W}$$

**EXAMPLE 3.33**

Find the value of $R_L$ in the network shown that will achieve maximum power transfer, and determine the value of the maximum power.

**SOLUTION**

Removing $R_L$ from the circuit of Fig. 3.98(a), we get the circuit of Fig. 3.98(b).

Applying **KVL clockwise** we get

$$-12 + 2 \times 10^3 i + 2V_x' = 0$$

Also $V_x' = 1 \times 10^3 i$

Hence, $-12 + 2 \times 10^3 i + 2 (1 \times 10^3 i) = 0$

$$i = \frac{12}{4 \times 10^3} = 3 \text{ mA}$$
Applying KVL to loop 1 kΩ → 2V_x → b − a, we get

\[ 1 \times 10^3 i + 2V_x' - V_t = 0 \]
\[ \Rightarrow \quad V_t = 1 \times 10^3 i + 2 \left(1 \times 10^3 i\right) = (1 \times 10^3 + 2 \times 10^3) i = 3 \times 10^3 \left(3 \times 10^{-3}\right) = 9 \text{ V} \]

To find \( R_t \), we need to find \( i_{sc} \). While finding \( i_{sc} \), none of the independent sources must be deactivated.

Applying KVL to mesh 1:

\[ -12 + V_x'' + 0 = 0 \]
\[ \Rightarrow \quad V_x'' = 12 \]
\[ \Rightarrow \quad 1 \times 10^3 i_1 = 12 \Rightarrow i_1 = 12 \text{ mA} \]

Applying KVL to mesh 2:

\[ 1 \times 10^3 i_2 + 2V_x'' = 0 \]
\[ \Rightarrow \quad 1 \times 10^3 i_2 = -24 \]
\[ i_2 = -24 \text{ mA} \]

Applying KCL at node a:

\[ i_{sc} = i_1 - i_2 = 12 + 24 = 36 \text{ mA} \]

Hence,

\[ R_t = \frac{V_t}{i_{sc}} = \frac{V_{oc}}{i_{sc}} = \frac{9}{36 \times 10^{-3}} = 250 \text{ Ω} \]

For maximum power transfer, \( R_t = R_L = 250 \text{ Ω} \). Thus, the Thevenin equivalent circuit with \( R_L \) is as shown in Fig 3.98 (c):

\[ i_T = \frac{9}{250 + 250} = \frac{9}{500} \text{ A} \]

\[ P_{\text{max}} = i_T^2 \times 250 = \left(\frac{9}{500}\right)^2 \times 250 = 81 \text{ mW} \]
EXAMPLE 3.34

The variable resistor $R_L$ in the circuit of Fig. 3.99 is adjusted until it absorbs maximum power from the circuit.

(a) Find the value of $R_L$.

(b) Find the maximum power.

![Circuit Diagram](image)

**SOLUTION**

Disconnecting the load resistor $R_L$ from the original circuit, we get the circuit shown in Fig. 3.99(a).

![Circuit Diagram](image)

**KCL at node $v_1$:**

\[
\frac{v_1 - 100}{2} + \frac{v_1 - 13i'_a}{5} + \frac{v_1 - v_2}{4} = 0
\]

(3.18)

**Constraint equations:**

\[
i'_a = \frac{100 - v_1}{2} \quad (3.19)
\]

\[
\frac{v_2 - v_1}{4} = i'_a \quad \text{(applying KCL at $v_2$)}
\]

(3.20)

\[
v'_a = v_1 - v_2 \quad (\text{potential across } 4 \Omega)
\]

(3.21)
From equations (3.20) and (3.21), we have

\[
\frac{v_2 - v_1}{4} = v_1 - v_2
\]
\[
\Rightarrow v_2 - v_1 = 4v_1 - 4v_2
\]
\[
\Rightarrow 5v_1 - 5v_2 = 0
\]
\[
\Rightarrow v_1 = v_2 \quad \text{(3.22)}
\]

Making use of equations (3.19) and (3.22) in (3.18), we get

\[
\frac{v_1 - 100}{2} + \frac{v_2 - 13(100 - v_1)}{5} + \frac{v_1 - v_1}{4} = 0
\]
\[
\Rightarrow 5(v_1 - 100) + 2\left[ v_1 - 13\left(\frac{100 - v_1}{2}\right) \right] = 0
\]
\[
\Rightarrow 5v_1 - 500 + 2v_1 - 13\times 100 + 13v_1 = 0
\]
\[
\Rightarrow 20v_1 = 1800
\]
\[
\Rightarrow v_1 = 90 \text{ Volts}
\]

Hence,
\[
v_1 = v_2 = v_1 = 90 \text{ Volts}
\]

We know that,
\[
R_t = \frac{v_{\text{oc}}}{i_{\text{sc}}} = \frac{v_t}{i_{\text{sc}}}
\]

The short circuit current is calculated using the circuit shown below:

Here
\[
i''_a = \frac{100 - v_1}{2}
\]

Applying KCL at node \(v_1\) :

\[
\frac{v_1 - 100}{2} + \frac{v_1 - 13i''_a}{5} + \frac{v_1 - 0}{4} = 0
\]
\[
\Rightarrow \frac{v_1 - 100}{2} + \frac{v_1 - 13}{5} + \frac{v_1}{4} = 0
\]
Solving we get \( v_1 = 80 \text{ volts} = v'_a \)

Applying KCL at node a :

\[
\frac{0 - v_1}{4} + i_{sc} = v''_a
\]

\[
\Rightarrow \quad i_{sc} = \frac{v_1}{4} + v''_a
\]

\[
= \frac{80}{4} + 80 = 100 \text{ A}
\]

Hence,

\[
R_L = \frac{v_{oc}}{i_{sc}} = \frac{v_t}{i_{sc}}
\]

\[
= \frac{90}{100} = 0.9 \ \Omega
\]

Hence for maximum power transfer,

\[
R_L = R_t = 0.9 \ \Omega
\]

The Thevenin equivalent circuit with \( R_L = 0.9 \ \Omega \) is as shown.

\[
i_t = \frac{90}{0.9 + 0.9} = \frac{90}{1.8}
\]

\[
P_{\text{max}} = i_t^2 \times 0.9 = \left( \frac{90}{1.8} \right)^2 \times 0.9 = 2250 \ \text{W}
\]

**EXAMPLE 3.35**

Refer to the circuit shown in Fig. 3.100 :

(a) Find the value of \( R_L \) for maximum power transfer.

(b) Find the maximum power that can be delivered to \( R_L \).
Removing the load resistor $R_L$, we get the circuit diagram shown in Fig. 3.100(a). Let us proceed to find $V_I$.

![Circuit Diagram](image)

**Constraint equation:**

$$i' = i_1 - i_3$$

**KVL clockwise to mesh 1:**

$$200 + 1(i_1 - i_2) + 20(i_1 - i_3) + 4i_1 = 0$$

$$\Rightarrow 25i_1 - i_2 - 20i_3 = -200$$

**KVL clockwise to mesh 2:**

$$14i' + 2(i_2 - i_3) + 1(i_2 - i_1) = 0$$

$$\Rightarrow 14(i_1 - i_3) + 2(i_2 - i_3) + 1(i_2 - i_1) = 0$$

$$\Rightarrow 13i_1 + 3i_2 - 16i_3 = 0$$

**KVL clockwise to mesh 3:**

$$2(i_3 - i_2) - 100 + 3i_3 + 20(i_3 - i_1) = 0$$

$$\Rightarrow -20i_1 - 2i_2 + 25i_3 = 100$$

Solving the mesh equations, we get

$$i_1 = -2.5A, i_3 = 5A$$

**Applying KVL clockwise to the path comprising of $a - b \rightarrow 20\,\Omega$, we get**

$$V_I - 20i' = 0$$

$$\Rightarrow V_I = 20i'$$

$$= 20(i_1 - i_3)$$

$$= 20(-2.5 - 5)$$

$$= -150\,\text{V}$$
Next step is to find \( R_t \).

\[
R_t = \frac{V_{oc}}{i_{sc}} = \frac{V_t}{i_{sc}}
\]

When terminals \( a - b \) are shorted, \( i''_a = 0 \). Hence, \( 14 i''_a \) is also zero.

**KVL clockwise to mesh 1:**

\[
200 + 1 (i_1 - i_2) + 4i_1 = 0
\]

\[
\Rightarrow 5i_1 - i_2 = -200
\]

**KVL clockwise to mesh 2:**

\[
2 (i_2 - i_3) + 1 (i_2 - i_1) = 0
\]

\[
\Rightarrow -i_1 + 3i_2 - 2i_3 = 0
\]

**KVL clockwise to mesh 3:**

\[
-100 + 3i_3 + 2 (i_3 - i_2) = 0
\]

\[
\Rightarrow -2i_2 + 5i_3 = 100
\]
Solving the mesh equations, we find that

\[ i_1 = -40 \text{A}, \quad i_3 = 20 \text{A}, \]

\[ \Rightarrow \quad i_{sc} = i_1 - i_3 = -60 \text{A} \]

\[ R_t = \frac{V_t}{i_{sc}} = \frac{-150}{-60} = 2.5 \Omega \]

For maximum power transfer, \( R_L = R_t = 2.5 \Omega \). The Thevenin equivalent circuit with \( R_L \) is as shown below:

For maximum power transfer, \( R_L = R_t = 2.5 \Omega \) . The Thevenin equivalent circuit with \( R_L \) is as shown below:

\[ P_{\text{max}} = i_1^2 R_L \]

\[ = \left[ \frac{150}{2.5 + 2.5} \right]^2 \times 2.5 \]

\[ = 2250 \text{ W} \]

**EXAMPLE 3.36**

A practical current source provides 10 W to a 250 Ω load and 20 W to an 80 Ω load. A resistance \( R_L \), with voltage \( v_L \) and current \( i_L \), is connected to it. Find the values of \( R_L, v_L \) and \( i_L \) if (a) \( v_L i_L \) is a maximum, (b) \( v_L \) is a maximum and (c) \( i_L \) is a maximum.

**SOLUTION**

Load current calculation:

10W to 250 Ω corresponds to \( i_L = \sqrt{\frac{10}{250}} \)

= 200 mA

20W to 80 Ω corresponds to \( i_L = \sqrt{\frac{20}{80}} \)

= 500 mA

Using the formula for division of current between two parallel branches:

\[ i_2 = \frac{i \times R_1}{R_1 + R_2} \]

In the present context,

\[ 0.2 = \frac{I_N R_N}{R_N + 250} \]  \[ \quad \text{(3.23)} \]

and

\[ 0.5 = \frac{I_N R_N}{R_N + 80} \]  \[ \quad \text{(3.24)} \]
Solving equations (3.23) and (3.24), we get

\[ I_N = 1.7 \text{ A} \]
\[ R_N = 33.33 \text{ Ω} \]

(a) If \( v_L i_L \) is maximum,

\[ R_L = R_N = 33.33 \text{ Ω} \]
\[ i_L = 1.7 \times \frac{33.33}{33.33 + 33.33} \]
\[ = 850 \text{ mA} \]
\[ v_L = i_L R_L = 850 \times 10^{-3} \times 33.33 \]
\[ = 28.33 \text{ V} \]

(b) \( v_L = I_N (R_N || R_L) \) is a maximum when \( R_N || R_L \) is a maximum, which occurs when \( R_L = \infty \).

Then, \( i_L = 0 \) and

\[ v_L = 1.7 \times R_N \]
\[ = 1.7 \times 33.33 \]
\[ = 56.66 \text{ V} \]

(c) \( i_L = \frac{I_N R_N}{R_N + R_L} \) is maximum when \( R_L = 0 \) Ω

\[ \Rightarrow i_L = 1.7 \text{ A} \text{ and } v_L = 0 \text{ V} \]

3.5 Sinusoidal steady state analysis using superposition, Thevenin and Norton equivalents

Circuits in the frequency domain with phasor currents and voltages and impedances are analogous to resistive circuits.

To begin with, let us consider the principle of superposition, which may be restated as follows:

For a linear circuit containing two or more independent sources, any circuit voltage or current may be calculated as the algebraic sum of all the individual currents or voltages caused by each independent source acting alone.
The superposition principle is particularly useful if a circuit has two or more sources acting at different frequencies. The circuit will have one set of impedance values at one frequency and a different set of impedance values at another frequency. Phasor responses corresponding to different frequencies cannot be superposed; only their corresponding sinusoids can be superposed. That is, when frequencies differ, the principle of superposition applies to the summing of time domain components, not phasors. Within a component, problem corresponding to a single frequency, however phasors may be superposed.

Thevenin and Norton equivalents in phasor circuits are found exactly in the same manner as described earlier for resistive circuits, except for the substitution of impedance $Z$ in place of resistance $R$ and subsequent use of complex arithmetic. The Thevenin and Norton equivalent circuits are shown in Fig. 3.101 and 3.102.

The Thevenin and Norton forms are equivalent if the relations

\[(a) \ Z_t = Z_N \] \[ (b) \ V_t = Z_N I_N \]

hold between the circuits.

A step by step procedure for finding the Thevenin equivalent circuit is as follows:

1. Identify a separate circuit portion of a total circuit.
2. Find $V_t = V_{oc}$ at the terminals.
3. (a) If the circuit contains only impedances and independent sources, then deactivate all the independent sources and then find $Z_t$ by using circuit reduction techniques.
   (b) If the circuit contains impedances, independent sources and dependent sources, then either short-circuit the terminals and determine $I_{sc}$ from which

\[ Z_t = \frac{V_{oc}}{I_{sc}} \]

or deactivate the independent sources, connect a voltage or current source at the terminals, and determine both $V$ and $I$ at the terminals from which

\[ Z_t = \frac{V}{I} \]

A step by step procedure for finding Norton equivalent circuit is as follows:

(i) Identify a separate circuit portion of the original circuit.
(ii) Short the terminals after separating a portion of the original circuit and find the current through the short circuit at the terminals, so that $I_N = I_{sc}$.
(iii) (a) If the circuit contains only impedances and independent sources, then deactivate all the independent sources and then find $Z_N = Z_t$ by using circuit reduction techniques.
   (b) If the circuit contains impedances, independent sources and one or more dependent sources, find the open-circuit voltage at the terminals, $V_{oc}$, so that $Z_N = Z_t = \frac{V_{oc}}{I_{sc}}$. 

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EXAMPLE 3.37

Find the Thevenin and Norton equivalent circuits at the terminals $a - b$ for the circuit in Fig. 3.103.

![Figure 3.103](image)

SOLUTION

As a first step in the analysis, let us find $V_t$.

Using the principle of current division,

$$I_o = \frac{8 \cdot 4}{8 + j10 - j5} = \frac{32}{8 + j5}$$

$$V_t = I_o (j10) = \frac{j320}{8 + j5} = 33.92 \angle 58^\circ \text{ V}$$

To find $Z_t$, deactivate all the independent sources. This results in a circuit diagram as shown in Fig. 3.103 (a).
The Thevenin equivalent circuit as viewed from the terminals $a - b$ is as shown in Fig 3.103(b). Performing source transformation on the Thevenin equivalent circuit, we get the Norton equivalent circuit.

\[
\begin{align*}
I_N &= \frac{V_t}{Z_t} = \frac{33.92 / 58^\circ}{10 / 26^\circ} \\
&= 3.392 / 32^\circ \text{ A} \\
Z_N &= Z_t = 10 / 26^\circ \text{ H}
\end{align*}
\]

**EXAMPLE 3.38**

Find $v_o$ using Thevenin’s theorem. Refer to the circuit shown in Fig. 3.104.

**SOLUTION**

Let us convert the circuit given in Fig. 3.104 into a frequency domain equivalent or phasor circuit (shown in Fig. 3.105(a)). $\omega = 1$

\[
\begin{align*}
10 \cos (t - 45^\circ) &\rightarrow 10 / -45^\circ \text{ V} \\
5 \sin (t + 30^\circ) &= 5 \cos (t - 60^\circ) &\rightarrow 5 / -60^\circ \text{ V} \\
L &= 1H \rightarrow j \omega L = j \times 1 \times 1 = j1 \Omega \\
C &= 1F \rightarrow \frac{1}{j \omega C} = \frac{1}{j \times 1 \times 1} = -j1 \Omega
\end{align*}
\]
Disconnecting the capacitor from the original circuit, we get the circuit shown in Fig. 3.105(b). This circuit is used for finding $V_t$.

**KCL at node a:**

$$V_t - \frac{10}{45^\circ} + \frac{V_t - 5}{60^\circ} = 0$$

Solving,

$$V_t = 4.97 \angle -40.54^\circ$$

To find $Z_t$ deactivate all the independent sources in Fig. 3.105(b). This results in a network as shown in Fig. 3.105(c):

$$Z_t = Z_{ab} = 3\Omega \| j1 \Omega$$

$$= \frac{j3}{3+j} = \frac{3}{10} (1+j3) \ \Omega$$

The Thevenin equivalent circuit along with the capacitor is as shown in Fig. 3.105(d).

$$V_o = \frac{V_t}{Z_t - j1} (-j1)$$

$$= \frac{4.97 \angle -40.54^\circ}{0.3(1+j3) - j1} (-j1)$$

$$= 15.73 \angle 247.9^\circ \ \text{V}$$

Hence, $v_o = 15.73 \cos (t + 247.9^\circ) \ \text{V}$
EXAMPLE 3.39

Find the Thevenin equivalent circuit of the circuit shown in Fig. 3.106.

**SOLUTION**

Since terminals $a - b$ are open,

$$V_a = I_a \times 10 = 20 \angle 0^\circ \text{ V}$$

*Applying KVL clockwise* for the mesh on the right hand side of the circuit, we get

$$-3V_a + 0(j10) + Vo_c - V_a = 0$$

$$Vo_c = 4V_a = 80 \angle 0^\circ \text{ V}$$

Let us transform the current source with $10 \Omega$ parallel resistance to a voltage source with $10 \Omega$ series resistance as shown in figure below:

To find $Z_t$, the independent voltage source is deactivated and a current source of $1 \text{ A}$ is connected at the terminals as shown below:

---

**Figure 3.106**
Applying KVL clockwise we get,

\[-V_a' - 3V_a' - j10I + V_o = 0\]
\[\Rightarrow -4V_a' - j10I + V_o = 0\]

Since \( V_a' = 10I \)
we get \(-40I - j10I = -V_o\)
Hence, \( Z_t = \frac{V_o}{I} = 40 + j10\Omega \)

Hence the Thevenin equivalent circuit is as shown in Fig 3.106(a):

**EXAMPLE 3.40**

Find the Thevenin and Norton equivalent circuits for the circuit shown in Fig. 3.107.

**SOLUTION**

The phasor equivalent circuit of Fig. 3.107 is shown in Fig. 3.108.

**KCL at node a:**

\[\frac{V_{oc} - 2V_{oc}}{j10} - 10 + \frac{V_{oc}}{-j5} = 0\]
\[\Rightarrow V_{oc} = -j \frac{100}{3} = \frac{100}{3} / -90^\circ \text{ V}\]
To find $\mathbf{I}_{sc}$, short the terminals $a - b$ of Fig. 3.108 as in Fig. 3.108(a).

Since $\mathbf{V}_{oc} = 0$, the above circuit takes the form shown in Fig. 3.108(b).

Thus, $\mathbf{I}_{sc} = 10 \angle 0^\circ \ A$

Hence, $\mathbf{Z}_t = \mathbf{V}_{oc} / \mathbf{I}_{sc} = \frac{100}{\frac{10}{10 \angle 0^\circ}} = \frac{10}{3} \angle -90^\circ \ \Omega$

The Thevenin equivalent and the Norton equivalent circuits are as shown below.

**EXAMPLE 3.41**

Find the Thevenin and Norton equivalent circuits in frequency domain for the network shown in Fig. 3.109.
SOLUTION

Let us find $V_a = V_{ab}$ using superpostion theorem.

(i) $V_{ab}$ due to $100/0^\circ$

$$I_1 = \frac{100/0^\circ}{-j300 + j100} = \frac{100}{-j200} \text{ A}$$

$$V_{eb1} = I_1(j100) = \frac{100}{-j200} (j100) = -50/0^\circ \text{ Volts}$$

(ii) $V_{ab}$ due to $100/90^\circ$
\[ I_2 = \frac{100 /90^\circ}{j100 - j300} \]
\[ V_{ab_2} = I_2 (-j300) \]
\[ = \frac{100 /90^\circ}{j100 - j300} (-j300) = j150 \text{ V} \]

Hence,
\[ V_t = V_{ab_1} + V_{ab_2} \]
\[ = -50 + j150 \]
\[ = 158.11 /108.43^\circ \text{ V} \]

To find \( Z_t \), deactivate all the independent sources.

\[ Z_t = j100 \Omega || - j300 \Omega \]
\[ = \frac{j100(-j300)}{j100 - j300} = j150 \Omega \]

Hence the Thevenin equivalent circuit is as shown in Fig. 3.109(a). Performing source transformation on the Thevenin equivalent circuit, we get the Norton equivalent circuit.

\[ I_N = \frac{V_t}{Z_t} = \frac{158.11 /108.43^\circ}{150 /90^\circ} = 1.054 /18.43^\circ \text{ A} \]
\[ Z_N = Z_t = j150 \Omega \]

The Norton equivalent circuit is as shown in Fig. 3.109(b).
3.6 Maximum power transfer theorem

We have earlier shown that for a resistive network, maximum power is transferred from a source to the load, when the load resistance is set equal to the Thevenin resistance with Thevenin equivalent source. Now we extend this result to the ac circuits.

In Fig. 3.110, the linear circuit is made up of impedances, independent and dependent sources. This linear circuit is replaced by its Thevenin equivalent circuit as shown in Fig. 3.111. The load impedance could be a model of an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $Z_t$ and the load impedance $Z_L$ are

$$Z_t = R_t + jX_t$$

and

$$Z_L = R_L + jX_L$$

The current through the load is

$$I = \frac{V_t}{Z_t + Z_L} = \frac{V_t}{(R_t + jX_t) + (R_L + jX_L)}$$

The phasors $I$ and $V_t$ are the maximum values. The corresponding $RMS$ values are obtained by dividing the maximum values by $\sqrt{2}$. Also, the $RMS$ value of phasor current flowing in the load must be taken for computing the average power delivered to the load. The average power delivered to the load is given by

$$P = \frac{1}{2} |I|^2 R_L$$

$$= \frac{|V_t|^2 R_L}{2 (R_t + R_L)^2 (X_t + X_L)^2}$$

(3.25)

Our idea is to adjust the load parameters $R_L$ and $X_L$ so that $P$ is maximum. To do this, we get $\frac{\partial P}{\partial R_L}$ and $\frac{\partial P}{\partial X_L}$ equal to zero.
\[ \frac{\partial P}{\partial X_L} = \frac{-|V_t|^2 R_L (X_t + X_L)}{\left[R_t + R_L\right]^2 + (X_t + X_L)^2} \]
\[ \frac{\partial P}{\partial R_L} = \frac{|V_t|^2 \left[(R_t + R_L)^2 + (X_t + X_L)^2 - 2R_L (R_t + R_L)\right]}{2 \left[(R_t + R_L)^2 + (X_t + X_L)^2\right]^2} \]

Setting \( \frac{\partial P}{\partial X_L} = 0 \) gives
\[ X_L = -X_t \quad (3.26) \]
and Setting \( \frac{\partial P}{\partial R_L} = 0 \) gives
\[ R_L = \sqrt{R_t^2 + (X_t + X_L)^2} \quad (3.27) \]

Combining equations (3.26) and (3.27), we can conclude that for maximum average power transfer, \( Z_L \) must be selected such that \( X_L = -X_t \) and \( R_L = R_t \). That is the maximum average power of a circuit with an impedance \( Z_t \) that is obtained when \( Z_L \) is set equal to complex conjugate of \( Z_t \).

Setting \( R_L = R_t \) and \( X_L = -X_t \) in equation (3.25), we get the maximum average power as
\[ P = \frac{|V_t|^2}{8R_t} \]

In a situation where the load is purely real, the condition for maximum power transfer is obtained by putting \( X_L = 0 \) in equation (3.27). That is,
\[ R_L = \sqrt{R_t^2 + X_t^2} = |Z_t| \]

Hence for maximum average power transfer to a purely resistive load, the load resistance is equal to the magnitude of Thevenin impedance.

### 3.6.1 Maximum Power Transfer When \( Z \) is Restricted

Maximum average power can be delivered to \( Z_L \) only if \( Z_L = Z_t^* \). There are few situations in which this is not possible. These situations are described below:

(i) \( R_L \) and \( X_L \) may be restricted to a limited range of values. With this restriction, choose \( X_L \) as close as possible to \(-X_t\) and then adjust \( R_L \) as close as possible to \( \sqrt{R_t^2 + (X_t + X_L)^2} \).

(ii) Magnitude of \( Z_L \) can be varied but its phase angle cannot be. Under this restriction, greatest amount of power is transferred to the load when \( |Z_L| = |Z_t| \).

\( Z_t^* \) is the complex conjugate of \( Z_t \).
EXAMPLE 3.42
Find the load impedance that transfers the maximum power to the load and determine the maximum power quantity obtained for the circuit shown in Fig. 3.112.

\[ Z_{L} = 5 + j6 \]

\[ I = \frac{10 / 0}{5 + 5} = 1 / 0^\circ \]

Hence, the maximum average power transferred to the load is

\[ P = \frac{1}{2} |I|^2 R_L \]

\[ = \frac{1}{2} (1)^2 \times 5 = 2.5 \text{ W} \]

EXAMPLE 3.43
Find the load impedance that transfers the maximum average power to the load and determine the maximum average power transferred to the load \( Z_L \) shown in Fig. 3.113.
The first step in the analysis is to find the Thevenin equivalent circuit by disconnecting the load \( Z_L \). This leads to a circuit diagram as shown in Fig. 3.114.

\[
V_t = V_{oc} = 4/0^\circ \times 3 = 12/0^\circ \text{ Volts (RMS)}
\]

To find \( Z_t \), let us deactivate all the independent sources of Fig. 3.114. This leads to a circuit diagram as shown in Fig 3.114 (a):

\[
Z_t = 3 + j4 \Omega
\]

The Thevenin equivalent circuit with \( Z_L \) is as shown in Fig. 3.115.

For maximum average power transfer to the load, \( Z_L = Z_t^* = 3 - j4 \).

\[
I_t = \frac{12/0^\circ}{3 + j4 + 3 - j4} = 2/0^\circ \text{ A (RMS)}
\]

Hence, maximum average power delivered to the load is

\[
P = |I_t|^2 R_L = 4(3) = 12 \text{ W}
\]

It may be noted that the scaling factor \( \frac{1}{2} \) is not taken since the phase current is already expressed by its RMS value.
EXAMPLE 3.44

Refer the circuit given in Fig. 3.116. Find the value of $R_L$ that will absorb the maximum average power.

![Figure 3.116](image)

SOLUTION

Disconnecting the load resistor $R_L$ from the original circuit diagram leads to a circuit diagram as shown in Fig. 3.117.

![Figure 3.117](image)

To find $Z_L$, let us deactivate all the independent sources present in Fig. 3.117 as shown in Fig 3.117 (a).

$$Z_L = \frac{(40 - j30) || j20}{j20 + 40 - j30} = (9.412 + j22.35) \, \Omega$$
The Value of $R_L$ that will absorb the maximum average power is

$$R_L = |Z_L| = \sqrt{(9.412)^2 + (22.35)^2}$$

$$= 24.25 \, \Omega$$

The Thevenin equivalent circuit with $R_L$ inserted is as shown in Fig 3.117 (b).

Maximum average power absorbed by $R_L$ is

$$P_{\text{max}} = \frac{1}{2} |I_t|^2 R_L$$

where

$$I_t = \frac{72.76 / 134^\circ}{(9.412 + j22.35 + 24.25)}$$

$$= 1.8 / 100.2^\circ \, A$$

$$\Rightarrow P_{\text{max}} = \frac{1}{2} (1.8)^2 \times 24.25$$

$$= 39.29 \, W$$

**EXAMPLE 3.45**

For the circuit of Fig. 3.118: (a) what is the value of $Z_L$ that will absorb the maximum average power? (b) what is the value of maximum power?

**SOLUTION**

Disconnecting $Z_L$ from the original circuit we get the circuit as shown in Fig. 3.119. The first step is to find $V_t$. 

![Figure 3.117 (a) Thevenin equivalent circuit](image-url-a)

![Figure 3.117 (b) Thevenin equivalent circuit](image-url-b)

![Figure 3.118](image-url-c)

![Figure 3.119](image-url-d)
\[ V_t = V_{oc} = I_1 (-j10) \]
\[ = \left[ \frac{120/0^\circ}{10 + j15 - j10} \right] (-j10) \]
\[ = 107.33/\!-116.57^\circ \text{ V} \]

The next step is to find \( Z_t \). This requires deactivating the independent voltage source of Fig. 3.119.

\[ Z_t = (10 + j15) || (-j10) \]
\[ = \frac{-j10 (10 + j15)}{-j10 + 10 + j15} \]
\[ = 8 - j14 \Omega \]

The value of \( Z_L \) for maximum average power absorbed is

\[ Z_L^* = 8 + j14 \Omega \]

The Thevenin equivalent circuit along with \( Z_L = 8 + j14 \Omega \) is as shown below:

\[ I_L = \frac{107.33/\!-116.57^\circ}{8 - j14 + 8 + j14} \]
\[ = \frac{107.33}{16} /\!-116.57^\circ \text{ A} \]

Hence,

\[ P_{\text{max}} = \frac{1}{2} |I_t|^2 R_L \]
\[ = \frac{1}{2} \left( \frac{107.33}{16} \right)^2 \times 8 \]
\[ = 180 \text{ Watts} \]
EXAMPLE 3.46

(a) For the circuit shown in Fig. 3.120, what is the value of \( Z_L \) that results in maximum average power that will be transferred to \( Z_L \)? What is the maximum power?

(b) Assume that the load resistance can be varied between 0 and 4000 \( \Omega \) and the capacitive reactance of the load can be varied between 0 and \(-2000\) \( \Omega \). What settings of \( R_L \) and \( X_C \) transfer the most average power to the load? What is the maximum average power that can be transferred under these conditions?

![Figure 3.120](image)

SOLUTION

(a) If there are no constraints on \( R_L \) and \( X_L \), the load impedance \( Z_L = Z_L^* = (3000 - j4000) \Omega \).

Since the voltage source is given in terms of its RMS value, the average maximum power delivered to the load is

\[
P_{\text{max}} = |I_t|^2 R_L
\]

where

\[
I_t = \frac{10 \angle 0^\circ}{3000 + j4000 + 3000 - j4000} = \frac{10}{2 \times 3000} A
\]

\[
\Rightarrow P_{\text{max}} = |I_t|^2 R_L = \frac{100}{4 \times (3000)^2} \times 3000 = 8.33 \text{ mW}
\]

(b) Since \( R_L \) and \( X_C \) are restricted, we first set \( X_C \) as close to \(-4000 \Omega \) as possible; hence \( X_C = -2000 \Omega \). Next we set \( R_L \) as close to \( \sqrt{R_t^2 + (X_C + X_L)^2} \) as possible.

Thus,

\[
R_L = \sqrt{3000^2 + (-2000 + 4000)^2} = 3605.55 \Omega
\]

Since \( R_L \) can be varied between 0 to 4000 \( \Omega \), we can set \( R_L \) to 3605.55 \( \Omega \). Hence \( Z_L \) is adjusted to a value

\[
Z_L = 3605.55 - j2000 \Omega
\]
The maximum average power delivered to the load is

\[ P_{\text{max}} = |I_t|^2 R_L \]

\[ = (1.4489 \times 10^{-3})^2 \times 3605.55 \]

\[ = 7.57 \text{ mW} \]

Note that this is less than the power that can be delivered if there are no constraints on \( R_L \) and \( X_L \).

**EXAMPLE 3.47**

A load impedance having a constant phase angle of \(-45^\circ\) is connected across the load terminals \( a \) and \( b \) in the circuit shown in Fig. 3.121. The magnitude of \( Z_L \) is varied until the average power delivered, which is the maximum possible under the given restriction.

(a) Specify \( Z_L \) in rectangular form.

(b) Calculate the maximum average power delivered under this condition.
This power is the maximum average power that can be delivered by this circuit to a load impedance whose angle is constant at $-45^\circ$. Again this quantity is less than the maximum power that could have been delivered if there is no restriction on $Z_L$. In example 3.46 part (a), we have shown that the maximum power that can be delivered without any restrictions on $Z_L$ is 8.33 mW.

### 3.7 Reciprocity theorem

The reciprocity theorem states that in a linear bilateral single source circuit, the ratio of excitation to response is constant when the positions of excitation and response are interchanged.

**Conditions to be met for the application of reciprocity theorem:**

(i) The circuit must have a single source.

(ii) Initial conditions are assumed to be absent in the circuit.

(iii) Dependent sources are excluded even if they are linear.

(iv) When the positions of source and response are interchanged, their directions should be marked same as in the original circuit.

**Example 3.48**

Find the current in 2 Ω resistor and hence verify reciprocity theorem.
**SOLUTION**

The circuit is redrawn with markings as shown in Fig 3.123 (a).

![Circuit Diagram](image)

Then,

\[ R_1 = (8^{-1} + 2^{-1})^{-1} = 1.6\Omega \]
\[ R_2 = 1.6 + 4 = 5.6\Omega \]
\[ R_3 = (5.6^{-1} + 4^{-1})^{-1} = 2.3333\Omega \]

Current supplied by the source = \( \frac{20}{4 + 2.3333} = 3.16 \text{ A} \)

Current in branch \( ab = I_{ab} = 3.16 \times \frac{4}{4 + 4 + 1.6} = 1.32 \text{ A} \)

Current in \( 2\Omega, I_1 = 1.32 \times \frac{8}{10} = 1.05 \text{ A} \)

**Verification using reciprocity theorem**

The circuit is redrawn by interchanging the position of excitation and response as shown in Fig 3.123 (b).

![Circuit Diagram](image)

Solving the equivalent resistances,

\[ R_4 = 2\Omega, \quad R_5 = 6\Omega, \quad R_6 = 3.4286\Omega \]

Now the current supplied by the source

\[ = \frac{20}{3.4286 + 2} = 3.6842\text{A} \]
Therefore,

\[ I_{cd} = 3.6842 \times \frac{8}{8 + 6} = 2.1053 \text{A} \]

\[ I_2 = \frac{2.1053}{2} = 1.05 \text{A} \]

As \( I_1 = I_2 = 1.05 \text{ A} \), reciprocity theorem is verified.

**EXAMPLE 3.49**

In the circuit shown in Fig. 3.124, find the current through 1.375 \( \Omega \) resistor and hence verify reciprocity theorem.

![Figure 3.124](image1)

**SOLUTION**

![Figure 3.125](image2)

**KVL clockwise for mesh 1**:

\[ 6.375I_1 - 2I_2 - 3I_3 = 0 \]

**KVL clockwise for mesh 2**:

\[ -2I_1 + 14I_2 - 10I_3 = 0 \]

**KVL clockwise for mesh 3**:

\[ -3I_1 - 10I_2 + 14I_3 = -10 \]
Putting the above three mesh equations in matrix form, we get

\[
\begin{bmatrix}
6.375 & -2 & -3 \\
-2 & 14 & -10 \\
-3 & -10 & 14 \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
-10 \\
\end{bmatrix}
\]

Using Cramer’s rule, we get

\[I_1 = -2A\]

Negative sign indicates that the assumed direction of current flow should have been the other way.

**Verification using reciprocity theorem:**

The circuit is redrawn by interchanging the positions of excitation and response. The new circuit is shown in Fig. 3.126.

The mesh equations in matrix form for the circuit shown in Fig. 3.126 is

\[
\begin{bmatrix}
6.375 & -2 & 3 \\
-2 & 14 & 10 \\
3 & 10 & 14 \\
\end{bmatrix}
\begin{bmatrix}
I'_1 \\
I'_2 \\
I'_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
0 \\
0 \\
\end{bmatrix}
\]

Using Cramer’s rule, we get

\[I'_3 = -2 A\]

Since \(I_1 = I'_3 = -2 A\), the reciprocity theorem is verified.

**EXAMPLE 3.50**

Find the current \(I_x\) in the \(j2 \, \Omega\) impedance and hence verify reciprocity theorem.
With reference to the Fig. 3.127, the current through $2\Omega$ impedance is found using series–parallel reduction techniques.

Total impedance of the circuit is

$$Z_T = (2 + j3) + (-j5) || (3 + j2)$$

$$= 2 + j3 + (-j5)(3 + j2)$$

$$= 2 + j3 - j5 + 3 + j2$$

$$= 6.537/19.36^\circ \ \Omega$$

The total current in the network is

$$I_T = \frac{36/0^\circ}{6.537/19.36^\circ}$$

$$= 5.507/-19.36^\circ \ \text{A}$$

Using the principle of current division, we find that

$$I_x = \frac{I_T(-j5)}{-j5 + 3 + j2}$$

$$= 6.49/-64.36^\circ \ \text{A}$$

**Verification of reciprocity theorem:**

The circuit is redrawn by changing the positions of excitation and response. This circuit is shown in Fig. 3.128.

Total impedance of the circuit shown in Fig. 3.128 is

$$Z'_T = (3 + j2) + (2 + j3) \ || (-j5)$$

$$= (3 + j2)(2 + j3)(-j5)$$

$$= 9.804/19.36^\circ \ \Omega$$

The total current in the circuit is

$$I'_T = \frac{36/0^\circ}{9.804/19.36^\circ} = 3.672/-19.36^\circ \ \text{A}$$

Using the principle of current division,

$$I_y = \frac{I'_T(-j5)}{-j5 + 2 + j3} = 6.49/-64.36^\circ \ \text{A}$$

It is found that $I_x = I_y$, thus verifying the reciprocity theorem.

**EXAMPLE 3.51**

Refer the circuit shown in Fig. 3.129. Find current through the ammeter, and hence verify reciprocity theorem.
SOLUTION

To find the current through the ammeter:

By inspection the loop equations for the circuit in Fig. 3.130 can be written in the matrix form as

$$\begin{bmatrix} 16 & -1 & -10 \\ -1 & 26 & -20 \\ -10 & -20 & 30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$$

Using Cramer’s rule, we get

$$I_1 = 4.6 \text{ A}$$
$$I_2 = 5.4 \text{ A}$$

Hence current through the ammeter = $$I_2 - I_1 = 5.4 - 4.6 = 0.8 \text{ A}$$.

Verification of reciprocity theorem:

The circuit is redrawn by interchanging the positions of excitation and response as shown in Fig. 3.131.

By inspection the loop equations for the circuit can be written in matrix form as

$$\begin{bmatrix} 15 & 0 & -10 \\ 0 & 25 & -20 \\ -10 & -20 & 31 \end{bmatrix} \begin{bmatrix} I'_1 \\ I'_2 \\ I'_3 \end{bmatrix} = \begin{bmatrix} -50 \\ 50 \\ 0 \end{bmatrix}$$

Using Cramer’s rule we get

$$I'_3 = 0.8 \text{ A}$$
Hence, current through the Ammeter = 0.8 A.

It is found from both the cases that the response is same. Hence the reciprocity theorem is verified.

**EXAMPLE 3.52**

Find current through 5 ohm resistor shown in Fig. 3.132 and hence verify reciprocity theorem.

![Figure 3.132](image)

**SOLUTION**

By inspection, we can write

\[
\begin{bmatrix}
12 & 0 & -2 \\
0 & 2 + j10 & -2 \\
-2 & -2 & 9
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
= 
\begin{bmatrix}
-20 \\
20 \\
0
\end{bmatrix}
\]

Using Cramer’s rule, we get

\[I_3 = 0.5376 / -126.25^\circ \text{ A}\]

Hence, current through 5 ohm resistor = \(0.5376 / -126.25^\circ \text{ A}\)

**Verification of reciprocity theorem:**

The original circuit is redrawn by interchanging the excitation and response as shown in Fig. 3.133.

![Figure 3.133](image)
Putting the three equations in matrix form, we get

\[
\begin{bmatrix}
12 & 0 & -2 \\
0 & 2 + j10 & -2 \\
-2 & -2 & 9 \\
\end{bmatrix}
\begin{bmatrix}
I'_1 \\
I'_2 \\
I'_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
20 \\
\end{bmatrix}
\]

Using Cramer’s rule, we get

\[
I'_1 = 0.3876 \, \angle -2.35 \, \text{A}
\]
\[
I'_2 = 0.456 \, \angle -78.9^\circ \, \text{A}
\]

Hence,

\[
I'_2 - I'_1 = -0.3179 - j0.4335
\]
\[
= 0.5376 \, \angle -126.25^\circ \, \text{A}
\]

The response in both cases remains the same. Thus verifying reciprocity theorem.

### 3.8 Millman’s theorem

It is possible to combine number of voltage sources or current sources into a single equivalent voltage or current source using Millman’s theorem. Hence, this theorem is quite useful in calculating the total current supplied to the load in a generating station by a number of generators connected in parallel across a busbar.

**Millman’s theorem states that if n number of generators having generated emfs \( E_1, E_2, \ldots, E_n \) and internal impedances \( Z_1, Z_2, \ldots, Z_n \), are connected in parallel, then the emfs and impedances can be combined to give a single equivalent emf of \( E \) with an internal impedance of equivalent value \( Z \).**

Where

\[
E = \frac{E_1 Y_1 + E_2 Y_2 + \ldots + E_n Y_n}{Y_1 + Y_2 + \ldots + Y_n}
\]

and

\[
Z = \frac{1}{Y_1 + Y_2 + \ldots + Y_n}
\]

where \( Y_1, Y_2, \ldots, Y_n \) are the admittances corresponding to the internal impedances \( Z_1, Z_2, \ldots, Z_n \), and are given by

\[
Y_1 = \frac{1}{Z_1}
\]
\[
Y_2 = \frac{1}{Z_2}
\]
\[
\vdots
\]
\[
Y_n = \frac{1}{Z_n}
\]

Fig. 3.134 shows a number of generators having emfs \( E_1, E_2, \ldots, E_n \) connected in parallel across the terminals \( x \) and \( y \). Also, \( Z_1, Z_2, \ldots, Z_n \) are the respective internal impedances of the generators.
The Thevenin equivalent circuit of Fig. 3.134 using Millman’s theorem is shown in Fig. 3.135. The nodal equation at \( x \) gives

\[
\frac{E_1 - E}{Z_1} + \frac{E_2 - E}{Z_2} + \cdots + \frac{E_n - E}{Z_n} = 0
\]

\[
\Rightarrow \left[ \frac{E_1}{Z_1} + \frac{E_2}{Z_2} + \cdots + \frac{E_n}{Z_n} \right] = E \left[ \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n} \right]
\]

\[
\Rightarrow E_1 Y_1 + E_2 Y_2 + \cdots + E_n Y_n = E \left[ \frac{1}{Z} \right]
\]

where \( Z = \) Equivalent internal impedance.

or

\[
E_1 Y_1 + E_2 Y_2 + \cdots + E_n Y_n = E Y
\]

\[
\Rightarrow E = \frac{E_1 Y_1 + E_2 Y_2 + \cdots + E_n Y_n}{Y}
\]

where

\[
Y = Y_1 + Y_2 + \cdots + Y_n
\]

and

\[
Z = \frac{1}{Y} = \frac{1}{Y_1 + Y_2 + \cdots + Y_n}
\]

**EXAMPLE 3.53**

Refer the circuit shown in Fig. 3.136. Find the current through 10 \( \Omega \) resistor using Millman’s theorem.
Using Millman’s theorem, the circuit shown in Fig. 3.136 is replaced by its Thevenin equivalent circuit across the terminals $PQ$ as shown in Fig. 3.137.

\[
E = \frac{E_1 Y_1 + E_2 Y_2 - E_3 Y_3}{Y_1 + Y_2 + Y_3}
\]
\[
= \frac{22 \left( \frac{1}{5} \right) + 48 \left( \frac{1}{12} \right) - 12 \left( \frac{1}{4} \right)}{\frac{1}{5} + \frac{1}{12} + \frac{1}{4}}
\]
\[
= 10.13 \text{ Volts}
\]
\[
R = \frac{1}{Y_1 + Y_2 + Y_3}
\]
\[
= \frac{1}{0.2 + 0.083 + 0.25}
\]
\[
= 1.88 \Omega
\]

Hence,
\[
I_L = \frac{E}{R + 10} = 0.853 \text{ A}
\]

**EXAMPLE 3.54**
Find the current through $(10 - j3)\Omega$ using Millman’s theorem. Refer Fig. 3.138.

The circuit shown in Fig. 3.138 is replaced by its Thevenin equivalent circuit as seen from the terminals, $A$ and $B$ using Millman’s theorem. Fig. 3.139 shows the Thevenin equivalent circuit along with $Z_L = 10 - j3 \Omega$. 

Hence,
\[
I_L = \frac{E}{R + 10} = 0.853 \text{ A}
\]
E = \frac{E_1 Y_1 + E_2 Y_2 - E_3 Y_3}{Y_1 + Y_2 + Y_3}

= \frac{100 \, 10^\circ \left( \frac{1}{5} \right) + 90 \, 45^\circ \left( \frac{1}{10} \right) + 80 \, 30^\circ \left( \frac{1}{20} \right)}{\frac{1}{5} + \frac{1}{10} + \frac{1}{20}}

= 88.49 /15.66^\circ \text{ V}

Z = R = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{\frac{1}{5} + \frac{1}{10} + \frac{1}{20}} = 2.86 \, \Omega

I = \frac{E}{Z + Z_L} = \frac{88.49 /15.66}{2.86 + 10 - j3} = 6.7 /28.79^\circ \, \text{ A}

Alternately,

E = \frac{E_1 Y_1 + E_2 Y_2 + E_3 Y_3 + E_4 Y_4}{Y_1 + Y_2 + Y_3 + Y_4}

= \frac{100 \times 5^{-1} + 90 \, 45^\circ \times 10^{-1} + 80 \, 30^\circ \times 20^{-1}}{5^{-1} + 10^{-1} + 20^{-1} + (10 - j3)^{-1}}

= 70 /12^\circ \text{ V}

Therefore,

I = \frac{70 /12^\circ}{10 - j3}

= 6.7 /28.8^\circ \, \text{ A}

\text{EXAMPLE 3.55}

Refer the circuit shown in Fig. 3.140. Use Millman’s theorem to find the current through \( (5 + j5) \, \Omega \) impedance.
**SOLUTION**

The original circuit is redrawn after performing source transformation of 5 A in parallel with 4 Ω resistor into an equivalent voltage source and is shown in Fig. 3.141.

Treating the branch $5 + j5\Omega$ as a branch with $E_o = 0V$,

$$E_{PQ} = \frac{E_1Y_1 + E_2Y_2 + E_3Y_3 + E_4Y_4}{Y_1 + Y_2 + Y_3 + Y_4}$$

$$= \frac{4 \times 2^{-1} + 8 \times 3^{-1} + 20 \times 4^{-1}}{2^{-1} + 3^{-1} + 4^{-1} + (5 - j5)^{-1}}$$

$$= 8.14 /4.83^\circ \text{ V}$$

Therefore current in $(5 + j5)\Omega$ is

$$I = \frac{8.14 /4.83^\circ}{5 + j5} = 1.15 /-40.2^\circ \text{ A}$$

Alternately

$E_{PQ}$ with $(5 + j5)$ open

$$E_{PQ} = \frac{E_1Y_1 + E_2Y_2 + E_3Y_3}{Y_1 + Y_2 + Y_3}$$

$$= \frac{4 \times 2^{-1} + 8 \times 3^{-1} + 20 \times 4^{-1}}{2^{-1} + 3^{-1} + 4^{-1}}$$

$$= 8.9231 \text{ V}$$
Equivalent resistance \( R = (2^{-1} + 3^{-1} + 4^{-1})^{-1} = 0.9231\Omega \)

Therefore current in \( (5 + j5)\Omega \) is

\[
I = \frac{8.9231}{0.9231 + 5 + j5} = 1.15 \angle -40.2^\circ \text{ A}
\]

**EXAMPLE 3.56**

Find the current through 2 \( \Omega \) resistor using Millman’s theorem. Refer the circuit shown in Fig. 3.142.

![Figure 3.142](image)

**SOLUTION**

The Thévenin equivalent circuit using Millman’s theorem for the given problem is as shown in Fig. 3.142(a).

\[
E = \frac{E_1 Y_1 + E_2 Y_2}{Y_1 + Y_2} = \frac{10/10^\circ \left[ \frac{1}{3 + j4} \right] + 25/90^\circ \left[ \frac{1}{5} \right]}{3 + j4 + \frac{1}{5}} = 10.06/97.12^\circ \text{ V}
\]

\[
Z = \frac{1}{Y_1 + Y_2} = \frac{1}{\frac{1}{3 + j4} + \frac{1}{5}} = 2.8/26.56^\circ \text{ } \Omega
\]

Hence,

\[
I_L = \frac{E}{Z + 2} = \frac{10.06/97.12^\circ}{2.8/26.56^\circ + 2} = 2.15/81.63^\circ \text{ A}
\]
Reinforcement problems

R.P. 3.1

Find the current in $2\ \Omega$ resistor connected between $A$ and $B$ by using superposition theorem.

\[ \text{Figure R.P. 3.1} \]

\[ \text{SOLUTION} \]

Fig. R.P. 3.1(a), shows the circuit with 2V-source acting alone (4V-source is shorted).

Resistance as viewed from 2V-source is $2 + R_1 \ \Omega$,

\[
R_1 = \left( \frac{3 \times 2}{5} + 1 \right) \parallel 12 = \frac{(1.2 + 1) \times 12}{14.2} = 1.8592 \ \Omega
\]

Hence, $I_a = \frac{2}{2 + 1.8592} = 0.5182 \ \text{A}$

Then, $I_b = I_a \times \frac{12}{12 + 1 + 1.2} = 0.438 \ \text{A}$

\[
I_1 = 0.438 \times \frac{3}{5} = 0.2628 \ \text{A}
\]

With 4V-source acting alone, the circuit is as shown in Fig. R.P. 3.1(b).
The resistance as seen by 4V-source is $3 + R_2$ where

$$R_2 = \left( \frac{2 \times 12}{14} + 1 \right) \| 2$$

$$= \frac{2.7143 \times 2}{4.7143} = 1.1551 \, \Omega$$

Hence,

$$I_0 = \frac{4}{3 + 1.1551} = 0.9635 \, A$$

Thus,

$$I_2 = \frac{I_0 \times 2.7143}{4.7143} = 0.555 \, A$$

Finally, applying the principle of superposition,

we get,

$$I_{AB} = I_1 + I_2$$

$$= 0.2628 + 0.555$$

$$= 0.818 \, A$$

**R.P. 3.2**

For the network shown in Fig. R.P. 3.2, apply superposition theorem and find the current $I$.

**SOLUTION**

Open the 5A-current source and retain the voltage source. The resulting network is as shown in Fig. R.P. 3.2(a).
The impedance as seen from the voltage source is

\[ Z = (4 - j2) + \frac{(8 + j10)(-j2)}{8 + j8} = 6.01 \angle -45^\circ \ \Omega \]

Hence,

\[ I_n = \frac{j20}{Z} = 3.328 \angle 135^\circ \ \text{A} \]

Next, short the voltage source and retain the current source. The resulting network is as shown in Fig. R.P. 3.2 (b). Here, \( I_3 = 5 \text{A} \). Applying KVL for mesh 1 and mesh 2, we get

\[ 8I_1 + (I_1 - 5)j10 + (I_1 - I_2)(-j2) = 0 \]

and

\[ (I_2 - I_1)(-j2) + (I_2 - 5)(-j2) + 4I_2 = 0 \]

Simplifying, we get

\[ (8 + j8)I_1 + j2I_2 = j50 \]

and

\[ j2I_1 + (4 - j4)I_2 = -j10 \]

Solving, we get

\[ I_b = \begin{bmatrix} 8 + j8 & j50 \\ j2 & -j10 \end{bmatrix} = 2.897 \angle -23.96^\circ \ \text{A} \]

Since, \( I_n \) and \( I_b \) are flowing in opposite directions, we have

\[ I = I_n - I_b = 6.1121 \angle 144.78^\circ \ \text{A} \]

Apply superposition theorem and find the voltage across 1 \( \Omega \) resistor. Refer the circuit shown in Fig. R.P. 3.3. Take \( v_1(t) = 5 \cos (t + 10^\circ) \) and \( i_2(t) = 3 \sin 2t \ \text{A} \).
To begin with let us assume $v_1(t)$ alone is acting. Accordingly, short 10V - source and open $i_2(t)$. The resulting phasor network is shown in Fig. R.P. 3.3(a).

\[
\begin{align*}
\omega &= 1 \text{ rad/sec} \\
5 \cos (t + 10^0) &\rightarrow 5 /10^0 \text{ V} \\
L_1 &= 1 \text{ H} \rightarrow j \omega L_1 = j1 \Omega \\
C_1 &= 1 \text{ F} \rightarrow \frac{1}{j \omega C_1} = -j1 \Omega \\
L_2 &= \frac{1}{2} \text{ H} \rightarrow j \omega L_2 = j\frac{1}{2} \Omega \\
C_2 &= \frac{1}{2} \text{ F} \rightarrow \frac{1}{j \omega C_2} = -j2 \Omega
\end{align*}
\]

\[\therefore \quad V_a = 5 /10^0 \text{ V} \]

\[v_a(t) = 5 \cos [t + 10^0]\]

Let us next assume that $i_2(t)$ alone is acting. The resulting network is shown in Fig. R.P. 3.3(b).

\[
\begin{align*}
\omega &= 2 \text{ rad/sec} \\
3 \sin 2t &\rightarrow 3 /0^0 \text{ A} \\
C_1 &= 1 \text{ F} \rightarrow \frac{1}{j \omega C_1} = -j\frac{1}{2} \Omega \\
L_1 &= 1 \text{ H} \rightarrow j \omega L_1 = j2 \Omega \\
C_2 &= \frac{1}{2} \text{ F} \rightarrow \frac{1}{j \omega C_2} = -j1 \Omega \\
L_2 &= \frac{1}{2} \text{ H} \rightarrow j \omega L_2 = j1 \Omega
\end{align*}
\]

\[V_b = 3 /0^0 \times \frac{j1.5}{1+j1.5} = 2.5 /33.7^0 \text{ A}\]

\[\Rightarrow \quad v_b(t) = 2.5 \sin [2t + 33.7^0] \text{ A}\]

Finally with 10V-source acting alone, the network is as shown in Fig. R.P. 3.3(c). Since $\omega = 0$, inductors are shorted and capacitors are opened.

Hence, $V_c = 10 \text{ V}$

Applying principle of superposition, we get.

\[v_2(t) = v_a(t) = v_b(t) + V_c\]

\[= 5 \cos (t + 10^0) + 2.5 \sin (2t + 33.7^0) + 10 \text{ Volts}\]
R.P 3.4

Calculate the current through the galvanometer for the Kelvin double bridge shown in Fig. R.P. 3.4. Use Thevenin’s theorem. Take the resistance of the galvanometer as 30 Ω.

Figure R.P. 3.4

SOLUTION

With $G$ being open, the resulting network is as shown in Fig. R.P. 3.4(a).

Figure 3.4(a)

$$V_A = I_1 \times 100 = \frac{10}{450} \times 100 = \frac{20}{9} \text{ V}$$

$$I_2 = \frac{10}{1.5 + \frac{45 \times 5}{50}} = 1.66, \quad I_B = \frac{I_2 \times 5}{45 + 5} = 0.1I_2$$

Hence,

$$V_B = I_2 \times 0.5 + I_B \times 10$$

$$= 2.5 \text{ V}$$

Thus,

$$V_{AB} = V_i = V_A - V_B = \frac{20}{9} - 2.5 = \frac{-5}{18} \text{ Volts}$$
To find $R_t$, short circuit the voltage source. The resulting network is as shown in Fig. R.P. 3.4(b).

Transforming the $\Delta$ between $B$, $E$ and $F$ into an equivalent $Y$, we get

$$R_B = \frac{35 \times 10}{50} = 7 \Omega, \quad R_E = \frac{35 \times 5}{50} = 3.5 \Omega, \quad R_F = \frac{5 \times 10}{50} = 1 \Omega$$

The reduced network after transformation is as shown in Fig. R.P. 3.4(c).

Hence,

$$R_{AB} = R_t = \frac{350 \times 100}{450} + \frac{4.5 \times 1.5}{6} + 7 = 85.903 \Omega$$

The Thevenin’s equivalent circuit as seen from $A$ and $B$ with $30 \Omega$ connected between $A$ and $B$ is as shown in Fig. R.P. 3.4(d).

$$I_G = \frac{-5}{85.903 + 30} = -2.4\text{mA}$$

Negative sign implies that the current flows from $B$ to $A$.

**R.P 3.5**

Find $I_x$ and $R$ so that the networks $N_1$ and $N_2$ shown in Fig. R.P. 3.5 are equivalent.
SOLUTION

Transforming the current source in $N_1$ into an equivalent voltage source, we get $N_3$ as shown in Fig. R.P. 3.5(a).

From $N_3$, we can write,

$$V - IR = I_S R$$  \hspace{1cm} (3.28)

From $N_2$ we can write,

$$I = -10I_a$$

Also from $N_2$,

$$V - 3 = -2I_a$$

$$\Rightarrow \quad V - 3 = -2 \left( \frac{-I}{10} \right)$$

$$\Rightarrow \quad V - 3 = \frac{I}{5}$$

$$\Rightarrow \quad V - \frac{I}{5} = 3$$  \hspace{1cm} (3.29)

For equivalence of $N_1$ and $N_2$, it is required that equations (3.28) and (3.29) must be same. Comparing these equations, we get

$$IR = \frac{I}{5} \quad \text{and} \quad I_S R = 3$$

$$R = 0.2 \ \Omega \quad \text{and} \quad I_S = \frac{3}{0.2} = 15 \ A$$

R.P. 3.6

Obtain the Norton’s equivalent of the network shown in Fig. R.P. 3.6.
SOLUTION
Terminals $a$ and $b$ are shorted. This results in a network as shown in Fig. R.P. 3.6(a)

Figure R.P. 3.6(a)

The mesh equations are

(i) \[ 9I_1 + 0I_2 - 6I_3 = 30 \]  
(ii) \[ 0I_1 + 25I_2 + 15I_3 = 30 \]  
(iii) \[ -6I_1 + 15I_2 + 23I_3 = 4V_X = 4(10I_2) \]
\[ \Rightarrow -6I_1 - 25I_2 + 23I_3 = 0 \]

Solving equations (3.30), (3.31) and (3.32), we get

\[ I_N = I_{sc} = I_3 = 1.4706 \text{A} \]

With terminals $ab$ open, $I_3 = 0$. The corresponding equations are

\[ 9I_1 = 30 \quad \text{and} \quad 25I_2 = 50 \]

Hence,

\[ I_1 = \frac{30}{9} \text{A} \quad \text{and} \quad I_2 = \frac{30}{25} \text{A} \]

Then,

\[ V_X = 10I_2 = 10 \times \frac{30}{25} = 12 \text{V} \]

Hence,

\[ V_t = V_{oc} = 15I_2 - 6I_1 - 4V_X = -50 \text{V} \]

Thus,

\[ R_t = \frac{V_{oc}}{I_{sc}} = \frac{-50}{1.4706} = -34 \text{Ω} \]

Hence, Norton’s equivalent circuit is as shown in Fig. R.P. 3.6(b).
For the network shown in Fig. R.P. 3.7, find the Thevenin’s equivalent to show that

\[ V_t = \frac{V_1}{2} \left( 1 + a + b - ab \right) \]

and

\[ Z_t = \frac{3 - b}{2} \]

SOLUTION

With \( xy \) open, \( I_1 = \frac{V_1 - aV_1}{2} \)

Hence,

\[ V_{oc} = V_t = aV_1 + I_1 + bI_1 \]
\[ = aV_1 + \frac{V_1 - aV_1}{2} + b \left( \frac{V_1 - aV_1}{2} \right) \]
\[ = \frac{V_1}{2} \left[ 1 + a + b - ab \right] \]

With \( xy \) shorted, the resulting network is as shown in Fig. R.P. 3.7(a).

Applying KVL equations, we get

(i) \( I_1 + (I_1 - I_2) = V_1 - aV_1 \)
\[ \Rightarrow 2I_1 - I_2 = V_1 - aV_1 \quad (3.33) \]

(ii) \( (I_2 - I_1) + I_2 = aV_1 + bI_1 \)
\[ \Rightarrow - (1 + b) I_1 + 2I_2 = aV_1 \quad (3.34) \]

Solving equations (3.33) and (3.34), we get

\[ I_{sc} = I_2 = \frac{V_1 (1 + a + b - ab)}{3 - b} \]
Hence,
\[ Z_l = \frac{V_{oc}}{I_{sc}} = \frac{V_1}{2} \frac{(1 + a + b - ab)}{V_1 (1 + a + b - ab)} (3 - b) = \frac{3 - b}{2} \]

**R.P 3.8**

Use Norton’s theorem to determine \( I \) in the network shown in Fig. R.P. 3.8. Resistance Values are in ohms.

![Figure R.P. 3.8](image_url)

**SOLUTION**

Let \( I_{AE} = x \) and \( I_{EF} = y \). Then by applying KCL at various junctions, the branch currents are marked as shown in Fig. R.P. 3.8(a). \( I_{sc} = 125 - x = I_{AB} \) on shorting \( A \) and \( B \).

Applying KVL to the loop \( ABCFEA \), we get
\[
0.04x + 0.01y + 0.02 (y - 20) + 0.03 (x - 105) = 0
\Rightarrow
0.07x + 0.03y = 3.55 \tag{3.35}
\]

Applying KVL to the loop \( EDCEF \), we get
\[
(x - y - 30) 0.03 + (x - y - 55) 0.02 - (y - 20) 0.02 - 0.01y = 0
\Rightarrow
0.05x - 0.08y = 1.6 \tag{3.36}
\]
Solving equations (3.35) and (3.36), we get

\[ x = 46.76 \text{ A} \]

Hence,

\[ I_{sc} = I_N = 120 - x \]
\[ = 78.24 \text{ A} \]

The circuit to calculate \( R_t \) is as shown in Fig. R.P. 3.8(b). All injected currents have been opened.

\[ R_t = 0.03 + 0.04 + \frac{0.03 \times 0.05}{0.08} \]
\[ = 0.08875 \Omega \]
The Norton’s equivalent network is as shown in Fig. R.P. 3.8(c).

\[ I = 78.24 \times \frac{0.08875}{0.08875 + 0.04} \]
\[ = 53.9\text{A} \]

**R.P. 3.9**

For the circuit shown in Fig. R.P. 3.9, find \( R \) such that the maximum power delivered to the load is 3 mW.

**SOLUTION**

For a resistive network, the maximum power delivered to the load is

\[ P_{\text{max}} = \frac{V_t^2}{4R_t} \]

The network with \( R_L \) removed is as shown in Fig. R.P. 3.9(a).

Let the opencircuit voltage between the terminals \( a \) and \( b \) be \( V_t \).

Then, applying KCL at node \( a \), we get

\[ \frac{V_t - 1}{R} + \frac{V_t - 2}{R} + \frac{V_t - 3}{R} = 0 \]

Simplifying we get

\[ V_t = 2 \text{ Volts} \]

With all voltage sources shorted, the resistance, \( R_t \) as viewed from the terminals, \( a \) and \( b \) is found as follows:

\[ \frac{1}{R_t} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} = \frac{3}{R} \]

\[ \Rightarrow \]
\[ R_t = \frac{R}{3} \Omega \]
Hence,

\[ P_{\text{max}} = \frac{2^2}{4 \times \frac{R}{3}} = \frac{3}{R} = 3 \times 10^{-3} \]

\[ \Rightarrow \quad R = 1 \, \text{k}\Omega \]

Refer Fig. R.P. 3.10, find \( X_1 \) and \( X_2 \) in terms of \( R_1 \) and \( R_2 \) to give maximum power dissipation in \( R_2 \).

The circuit for finding \( Z_t \) is as shown in Figure R.P. 3.10(a).

\[ Z_t = \frac{R_1 (jX_1)}{R_1 + jX_1} \]

\[ = \frac{R_1 X_1^2 + j R_1^2 X_1}{R_1^2 + X_1^2} \]

For maximum power transfer,

\[ Z_L = Z_t^* \]

\[ \Rightarrow \quad R_2 + j X_2 = \frac{R_1 X_1^2}{R_1^2 + X_1^2} - j \frac{R_1^2 X_1}{R_1^2 + X_1^2} \]

Hence,

\[ R_2 = \frac{R_1 X_1^2}{R_1^2 + X_1^2} \]

\[ \Rightarrow \quad X_1 = \pm R_1 \sqrt{\frac{R_2}{R_1 - R_2}} \quad (3.37) \]

\[ X_2 = - \frac{R_1^2 X_1}{R_1^2 + X_1^2} \quad (3.38) \]

Substituting equation (3.37) in equation (3.38) and simplifying, we get

\[ X_2 = \sqrt{R_2 (R_1 - R_2)} \]
**Exercise Problems**

**E.P 3.1**
Find $i_x$ for the circuit shown in Fig. E.P. 3.1 by using principle of superposition.

![Figure E.P. 3.1](image1)

Ans: $i_x = -\frac{1}{4} \text{ A}$

**E.P 3.2**
Find the current through branch PQ using superposition theorem.

![Figure E.P. 3.2](image2)

Ans: 1.0625 A

**E.P 3.3**
Find the current through 15 ohm resistor using superposition theorem.

![Figure E.P. 3.3](image3)

Ans: 0.3826 A
E.P 3.4

Find the current through $3 + j4$ $\Omega$ using superposition theorem.

![Figure E.P. 3.4](image)

Ans: $8.3/85.3^\circ$ A

E.P 3.5

Find the current through $I_x$ using superposition theorem.

![Figure E.P. 3.5](image)

Ans: $3.07/-163.12^\circ$ A

E.P 3.6

Determine the current through 1 $\Omega$ resistor using superposition theorem.

![Figure E.P. 3.6](image)

Ans: $0.406$ A
E.P 3.7
Obtain the Thevenin equivalent circuit at terminals $a - b$ of the network shown in Fig. E.P. 3.7.

![Network Diagram](image)

**Ans:** $V_t = 6.29\, \text{V}, R_t = 9.43\, \Omega$

E.P 3.8
Find the Thevenin equivalent circuit at terminals $x - y$ of the circuit shown in Fig. E.P. 3.8.

![Network Diagram](image)

**Ans:** $V_t = 0.192\angle -43.4^\circ\, \text{V}, Z_t = 88.7/11.55^\circ\, \Omega$

E.P 3.9
Find the Thevenin equivalent of the network shown in Fig. E.P. 3.9.

![Network Diagram](image)

**Ans:** $V_t = 17.14\, \text{volts}, R_t = 4\, \Omega$
E.P. 3.10
Find the Thevenin equivalent circuit across $a - b$. Refer Fig. E.P. 3.10.

\[ V_T = -30 \text{ V}, \quad R_T = 10 \text{ k}\Omega \]

Figure E.P. 3.10

E.P. 3.11
Find the Thevenin equivalent circuit across $a - b$ for the network shown in Fig. E.P. 3.11.

Ans: Verify your result with other methods.

Figure E.P. 3.11

E.P. 3.12
Find the current through 20 ohm resistor using Norton equivalent.

\[ I_N = 4.36 \text{ A}, \quad R_N = R_T = 8.8 \text{ \Omega}, \quad I_L = 1.33 \text{ A} \]

Figure E.P. 3.12
E.P 3.13

Find the current in 10 ohm resistor using Norton’s theorem.

\[ I_N = 4 \, \text{A}, \quad R_t = R_N = \frac{100}{7} \, \Omega, \quad I_L = -0.5 \, \text{A} \]

E.P 3.14

Find the Norton equivalent circuit between the terminals \( a \)–\( b \) for the network shown in Fig. E.P. 3.14.

\[ I_N = 4.98310 \, \exp(-5.71^\circ) \, \text{A}, \quad Z_t = Z_N = 3.6 / 23.1^\circ \, \Omega \]

E.P 3.15

Determine the Norton equivalent circuit across the terminals \( P \)–\( Q \) for the network shown in Fig. E.P. 3.15.

\[ I_N = 5 \, \text{A}, \quad R_N = R_t = 6 \, \Omega \]
E.P 3.16
Find the Norton equivalent of the network shown in Fig. E.P. 3.16.

\[ \text{Ans: } I_N = 8.87 \text{ A, } R_N = R_t = 43.89 \text{ } \Omega \]

E.P 3.17
Determine the value of \( R_L \) for maximum power transfer and also find the maximum power transferred.

\[ \text{Ans: } R_L = 1.92 \text{ } \Omega, \ P_{\text{max}} = 4.67 \text{ W} \]

E.P 3.18
Calculate the value of \( Z_L \) for maximum power transfer and also calculate the maximum power.

\[ \text{Ans: } Z_L = (7.97 + j2.16) \text{ } \Omega, \ P_{\text{max}} = 0.36 \text{ W} \]
**E.P 3.19**

Determine the value of $R_L$ for maximum power transfer and also calculate the value of maximum power.

\[
\begin{align*}
\text{Ans: } & R_L = 5.44 \, \Omega, \quad P_{\text{max}} = 2.94 \, \text{W} \\
\end{align*}
\]

**E.P 3.20**

Determine the value of $Z_L$ for maximum power transfer. What is the value of maximum power?

\[
\begin{align*}
\text{Ans: } & Z_L = 4.23 + j1.15 \, \Omega, \quad P_{\text{max}} = 5.68 \, \text{Watts} \\
\end{align*}
\]

**E.P 3.21**

Obtain the Norton equivalent across $x - y$.

\[
\begin{align*}
\text{Ans: } & I_N = I_{SC} = 7.35 \, \text{A}, \quad R_L = R_N = 1.52 \, \Omega \\
\end{align*}
\]

**E.P 3.22**

Find the Norton equivalent circuit at terminals $a - b$ of the network shown in Fig. E.P. 3.22.
Circuit Theorems

Ans:$ I_N = 1.05 \angle 251.6^\circ \text{ A, } Z_t = Z_N = 10.6 \angle 45^\circ \text{ } \Omega$

E.P 3.23
Find the Norton equivalent across the terminals $X - Y$ of the network shown in Fig. E.P. 3.23.

Ans: $I_N = 7 \text{ A, } Z_t = 8.19 \angle -55^\circ \text{ } \Omega$

E.P 3.24
Determine the current through 10 ohm resistor using Norton’s theorem.

Ans: 0.15A
Determine the current $I$ using Norton’s theorem.

**Ans:** Verify your result with other methods.

Find $V_x$ in the circuit shown in Fig. E.P. 3.26 and hence verify reciprocity theorem.

**Ans:** $V_x = 9.28 \angle 21.81^\circ$ V

Find $V_x$ in the circuit shown in Fig. E.P. 3.27 and hence verify reciprocity theorem.

**Ans:** $V_x = 10.23$ Volts
E.P. 3.28

Find the current $i_{x}$ in the bridge circuit and hence verify reciprocity theorem.

$$i_{x} = 0.031 \text{ A}$$

E.P. 3.29

Find the current through 4 ohm resistor using Millman’s theorem.

$$I = 2.05 \text{ A}$$

E.P. 3.30

Find the current through the impedance of $(10 + j10) \Omega$ using Millman’s theorem.

$$3.384 /12.6^\circ \text{ A}$$
Using Millman’s theorem, find the current flowing through the impedance of \((4 + j3) \, \Omega\).

Ans: \(3.64/15.23^\circ\) A